AVERAGE LIKELIHOOD METHODS FOR CODE DIVISION MULTIPLE ACCESS (CDMA)

MAY 2014

FINAL TECHNICAL REPORT

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED

STINFO COPY

AIR FORCE RESEARCH LABORATORY
INFORMATION DIRECTORATE

■ **AIR FORCE MATERIEL COMMAND** ■ **UNITED STATES AIR FORCE** ■ **ROME, NY 13441**

NOTICE AND SIGNATURE PAGE

AFRL-RI-RS-TR-2014-093   HAS BEEN REVIEWED AND IS APPROVED FOR PUBLICATION IN ACCORDANCE WITH ASSIGNED DISTRIBUTION STATEMENT.

FOR THE DIRECTOR:

  // SIGNED //   // SIGNED //

BRENT HOLMES   WARREN H. DEBANY, JR

Chief, Cyber Operations Branch   Technical Advisor, Information
                                    Exploitation and Operations Division
                              Information Directorate

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From - To)*
May 2014	FINAL TECHNICAL REPORT	OCT 2011 – OCT 2013

4. TITLE AND SUBTITLE

AVERAGE LIKELIHOOD METHODS FOR CODE DIVISION MULTIPLE ACCESS (CDMA)

5a. CONTRACT NUMBER
IN-HOUSE

5b. GRANT NUMBER
N/A

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

Alfredo Vega Irizarry

5d. PROJECT NUMBER
GCIH

5e. TASK NUMBER
OM

5f. WORK UNIT NUMBER
MC

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory/Information Directorate
Rome Research Site/RIGB
525 Brooks Road
Rome NY 13441-4505

8. PERFORMING ORGANIZATION REPORT NUMBER

N/A

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Research Laboratory/Information Directorate
Rome Research Site/RIGB
525 Brooks Road
Rome NY 13441-4505

10. SPONSOR/MONITOR'S ACRONYM(S)
AFRL/RI

11. SPONSORING/MONITORING AGENCY REPORT NUMBER
AFRL-RI-RS-TR-2014-093

12. DISTRIBUTION AVAILABILITY STATEMENT
Approved for Public Release; Distribution Unlimited. PA# 88ABW-2014-1899
Date Cleared: 23 Apr 2014

13. SUPPLEMENTARY NOTES

14. ABSTRACT
This final report summarizes the most significant findings of the work done under the in-house effort titled "Optimal Methods for Modulation Classification" on average likelihood methods. The development began with the exact calculation of the likelihood function using symbolic algebra software. After investigating its properties, it was possible to develop formulas for more general cases. The solution of the likelihood function depends on a joint probability of the spreading code coefficients/elements. Simplification of the likelihood was achieved using the invariance properties of the Total Squared Correlation to permutations. The average likelihood can be expressed as sum of a product of hyperbolic cosine functions. Most these terms vanish after adjusting precision parameter from the joint probability of the code matrix. For a full loaded CDMA signal, the average likelihood depends exclusively on feature vectors or the absolute value of the sum of the columns of the matrix with the lowest Total Square Correlation. The ROC curves were plotted for characterizing the performance of the classifier.

15. SUBJECT TERMS

Classification, Detection, Modulation, LRT, Expectation Maximization

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**	UU	62	**Alfredo Vega Irrizarry**
U	U	U			19b. TELEPHONE NUMBER *(Include area code)* **N/A**

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

Table of Contents

List of Figures

List of Tables

Summary

This report presents the development of a likelihood ratio test for synchronous and balanced CDMA signals. The hypothesis associated with a binary test is the code length and number of user. The approach incorporates a statistical dependency of the spreading matrix elements when computing the expectation over the spreading matrix elements. The dependency is a discrete Gaussian probability function that depends on a modified Total Square Correlation. The resulting average likelihood is a product of terms that depends on the matrix that achieves the lowest Total Square Correlation. If the set of signals is constrained to Hadamard matrices using Sylvester's construction, the likelihood function can be simplified further, allowing the classification over code lengths in the range of 2^2 to 2^{13} and possibly higher.

Keywords: DS/CDMA signals, classification, balanced CDMA load, synchronous CDMA, decision theory, average likelihood ratio test, Total Square Correlation

Acknowledgments

1. Introduction

Modulation classification is an important step in the characterization of signals and the recovery of the information content. In a non-cooperative scenario, an eavesdropper attempts to gain information on how the signal has been synchronized and modulated before trying to recover the information content. A second scenario could be the case of a smart receiver with software defined radio that reconfigures itself for various modulation types, managing the allocation of users and adjusting the code length for efficient utilization of the bandwidth.

The problem of signal classification has been widely studied for MPSK and QAM signals using a decision theoretic approach. Surprisingly there is a lack of research on the area of classifying CDMA signals using average likelihood methods. [1]

Traditional approaches for signal classification include heuristics, feature-based classification and decision theory. The development of decision theoretic approaches must result, by virtue of the theory, in an optimal classifier that guarantees the lowest probability of error in classification. If there were such classifier that could perform better, then we would either omit an important piece of information in our assumptions or have discovered a novel approach that performs better than the current state of the art in the classification theory. The idea of developing average likelihood methods in CDMA sounds attractive because it could also result in better estimation methods. In a future research, we would consider the ambitious goal of estimating the CDMA spreading matrix or even estimating the information contents (multi-user detection) without knowledge of the spreading matrix.

The objective of this research is to develop a decision theoretic approach for modulation classification of CDMA signals. This approach will be based on the classical average likelihood ratio test (ALRT). We begin this classification problem by finding the size of the spreading matrix that generated the DS-CDMA signal. As the number of unknown variables grows, the averaging process becomes an extremely complex task. In the multiuser detection, a closely related problem, authors use Singular Value Decomposition (SVD) to estimate the unknown variables and do estimation based on a generalized likelihood approach. [2] These approaches are suboptimal by nature.

The benefits of the theory sound promising; however, there is a high risk due to the difficulty of averaging over a large number of parameters and obtaining an exact analytical solution. During the first year of the research, the risk was mitigated by concentrating in the development an empirical likelihood function for CDMA. The understanding was possible by using symbol algebra software and attempting to solve simple cases of CDMA. This knowledge was collected in a previous Interim Technical Report AFRL-RI-RS-TR-2013-058. Most of the previous work was superseded by the research done during the second year.

In the second part of the research, an effort was made to improve and provide a mathematical foundation for deriving an empirical likelihood function found during the first year. This includes extending the likelihood function to higher dimensions of the spreading matrix. A cluster of 48 computers was used calculate the likelihood to code length dimensions 5 and 6. The work required translating an empirically derived Matlab code (see Appendix G) to a Multiple Programming Interface in C++. Unfortunately the resulting approach was computationally overwhelming for code lengths above 6. However, the code length cases of 5 and 6 provided more evidence of the dependency of the likelihood on matrices that achieve the lowest Total Square Correlation. At this point, it was clear that the average likelihood function for CDMA is a sum of products of two main terms, that is, some coefficients that depended on the Total Square Correlation and a product of hyperbolic cosine functions. This product came from an averaging of exponentials as shown in Appendix E. But using this product would require partitioning the set of all possible combination of the spreading matrix in carefully selected subsets as shown in Appendix C. Another interesting observation was made. Most of the coefficients of the likelihood function are negligible for computing the likelihood as shown in Figures 3 and 4. This property reduced greatly the complexity of the average likelihood. The final task was to provide experimental evidence that validates our development. A classification using Receiver Operating Characteristic (ROC) curves revealed that the approach taken in this research was able to produce desirable results.

This report presents a mathematical development of the CDMA average likelihood function. It is divided in the following parts: a review of existing techniques, a description of the model, the problem and assumptions, the formulation of the likelihood function and its simplification. The metric of performance used in the simulations is the ROC curves which provide a visual characterization of the new classification rules. The results appear to support the validity of the newly generated classification rules.

2. Theoretical Background

The classification of DS/CDMA signals should not be confused with the problem of multiuser detection. The multiuser detection deals with the estimation of information symbols in a CDMA system. This can be done in the absence of the code that generated the signal. The different approaches try to gain information of the spreading matrix by using Singular Value Decomposition (SVD) methods. A common metric is the bit error rate (BER). On the other hand, the goal of modulation classification is the selection of a modulation scheme over the others. The classifier can be characterized using Receiver Operating Characteristic (ROC) curves or by generating confusion matrices.

As an estimation problem, the number of users or information symbols is obtained by maximizing a conditional likelihood function. Common assumptions in this problem are the estimation under Additive White Gaussian Noise (AGWN) and the full synchronization of the CDMA signal. This includes frame, chip and phase synchronized system [3]. Maximizing the likelihood function is prohibitively

complex due to the large amount of unknown parameters that are required to be estimated.

There is a lack of likelihood methods for modulation classification. A survey on classification methods discusses only traditional modulation schemes such as MPSK and QPSK methods [4]. A more recent paper highlights the need for modulation classification [5], but none of the methods directly addresses the classification of CDMA signals. A feature based classification technique is shown presented in [6]. This approach classifies the signal by its code length. It uses cyclostationary features and a neural network for classification. The method requires training and there is no guarantee of achieving an optimal performance.

3. Synchronous, Balanced CDMA Average Likelihood

3.1 Model

The goal is to classify CDMA signals by their code length L and number of users U in the presence of Additive White Gaussian Noise (AWGN). A CDMA signal $\vec{x}$ of size $L \times 1$ (see Eq. 1) is generated by a spreading matrix C of size $L \times U$, and the information symbol vector $\vec{b}$ of size $U \times 1$. For simplicity, we assume that the signal the energy per symbol E of each user is constant, i.e., a balanced load. Developing the case of unbalanced load is possible, but not considered in this report due to time constraints.

$$\vec{x} = \sqrt{E/L} \cdot C\vec{b}$$

(1)

The spreading matrix and the information vector are binary antipodal. The spreading matrix is a block matrix composed of column vectors $\vec{c}_{*,i}$ with low cross-correlation. A metric that characterizes the spreading matrix is known as the Total Square Correlation (TSC). A modified TSC is defined in this paper by [7]

$$\tau(C) \triangleq \sum_{\substack{i=0}}^{U-1} \sum_{\substack{j=0 \\ j \neq i}}^{U-1} \left| \langle \vec{c}_{*,i}, \vec{c}_{*,j} \rangle \right|^2.$$

(2)

An alternative definition of [Eq. 2] for an antipodal matrix can be expressed in terms of the Frobenius Norm as:

$$\tau(C) \triangleq \|C^T C\|_F^2 - L \cdot U.$$

(3)

The noise vector $\vec{n}$ has zero mean, and covariance matrix $\Sigma = N_0/2 \cdot I$, where I is an $L \times L$ identity matrix. The receiver has no knowledge of the spreading code, the code length or the number of users. The channel is modeled by:

$$\vec{y} = \vec{x} + \vec{n}.$$

$$(4)$$

This model makes the following assumptions: chip synchronization and knowledge of the beginning of the sequence. For simplicity, our approach will use similar assumptions to those used in multiuser detection, i.e., chip synchronization and frame synchronization. The assumptions represent a scenario where the eavesdropper has information of the preamble and knows the beginning of the transmission.

The conditional likelihood function is given by:

$$\lambda\left(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\right) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{(\vec{y}-\vec{x})^H \Sigma^{-1}(\vec{y}-\vec{x})}{2}}$$

$$(5)$$

The likelihood is conditioned on the hypothesis $\mathcal{H} = \{L, U\}$, the spreading matrix C, and the information vector $\vec{b}$ according to (5). The likelihood has made the assumption of the code length L and frame period. The assumption must be tested against other possible assumptions in order to make a determination of the code length.

The assumption of frame synchronization is not a critical restriction when constructing an average likelihood function. Additional averaging will be sufficient to remove the assumption. Once it is removed, it will represent a more realistic scenario where the eavesdropper does not know the beginning of the sequence.

Under the assumption of no frame synchronization, our conditional likelihood is given by:

$$\lambda\left(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}, \epsilon\right) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{(\vec{y}_\epsilon-\vec{x})^H \Sigma^{-1}(\vec{y}_\epsilon-\vec{x})}{2}}$$

$$(6)$$

where $\vec{y}_\epsilon$ represents the received vector that is delayed ϵ chips.

3.2 Average Likelihood Function

The construction of our likelihood function will be expressed as a function of the energy per chip $E_c = E/L$ and noise power N_0. Equation (7) introduces new variables for convenience. These are the energy-to-noise ratio per chip γ, the unnormalized signal $\vec{s}$, and the correlator output $\vec{r}$ respectively:

$$\gamma = \frac{E}{N_0 L}, \quad \vec{s} = C\vec{b}, \quad \vec{r} = \frac{\vec{y}}{\sqrt{N_0/2}}.$$

$$(7)$$

The conditional likelihood function of (5) becomes:

$$\lambda(\vec{r}|\mathcal{H}, C, \vec{b}) = \left(\frac{e^{-\langle \vec{r}, \vec{r} \rangle}}{(\pi N_0)^L}\right)^{1/2} e^{\sqrt{2\gamma}\langle \vec{r}, \vec{s} \rangle - \gamma \langle \vec{s}, \vec{s} \rangle}.$$

<div align="right">(8)</div>

(See Appendix A) Unlike the MPSK likelihood [1], all terms in (8) are kept because of their dependency on the hypothesis.

Up to this point, the discussion has not added any new development. The new contribution is contained in the following premise. The average of the conditional likelihood is an interesting case of dependency between the elements of the spreading matrix. It is unlikely that a random selection of matrix elements would result in a low correlation matrix. Therefore, we must selectively choose the spreading matrices based on some code-design metric. This approach constructs a Gaussian-like discrete probability distribution based on our definition of the TSC. The probability of choosing a suitable spreading matrix with low-correlation is given by:

$$P(C) \triangleq \frac{e^{-\beta \cdot \tau(C)}}{\sum_{\text{All } c_{i,j}} e^{-\beta \cdot \tau(C)}}.$$

<div align="right">(9)</div>

For a perfectly uncorrelated code such as Walsh codes, the TSC metric is zero and the probability achieves a maximum value. This probability has multiple maxima. Any sign inversion or permutation of rows and column of matrix C must have the same TSC value and thus multiple maxima are expected. (See Appendix C)

The parameter β is called precision and it is half the inverse of the variance of this particular distribution. [8] Consider $p(\mathcal{H})$ being the probability of a hypothesis $\mathcal{H}$. The detection of $\mathcal{H}$ would depend on $p(\mathcal{H})$ integrated from a threshold η to infinity. The resulting probability of detection as a function of η is a sigmoid function called error function. If β is the precision of $p(\mathcal{H})$ and this variance is allowed going to infinity, then the slope of the sigmoid becomes steeper to a point when the probability of detection is either zero or one. This adjustment in precision results in a hard decision scheme. Later on, we will investigate the case when the precision of this probability approaches to infinity.   Averaging over the code coefficients eliminates the dependency of the likelihood on the spreading matrix as follows:

$$\lambda(\vec{r}|\mathcal{H}, \vec{b}) = \left(\frac{e^{-\langle \vec{r}, \vec{r} \rangle}}{(\pi N_0)^L}\right)^{1/2} \sum_{\text{All } c_{i,j}} P(C)\, e^{\sqrt{2\gamma}\langle \vec{r}, \vec{s} \rangle - \gamma \langle \vec{s}, \vec{s} \rangle}.$$

<div align="right">(10)</div>

Finally, the dependency on $\vec{b}$ is removed by averaging over uniformly distributed information symbols as given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^L}\right)^{1/2} \sum_{All\ b_j} \frac{1}{2^U} \sum_{All\ c_{i,j}} P(C)\, e^{\sqrt{2\gamma}\langle\vec{r},\vec{s}\rangle - \gamma\langle\vec{s},\vec{s}\rangle}.$$

$$(11)$$

3.3 Simplification

Averaging process for hypothesis $\mathcal{H} = \{2,2\}$ has 8 unknown variables and produces $2^8 = 64$ exponential terms. In general, the number of possible combinations involved in the averaging process increases exponentially:

$$2^{L \cdot U + U}.$$

$$(12)$$

For a 4x4 spreading code, the number of averaging computations is: $2^{4^2+4} = 2^{20}$.

The likelihood expression in (11) is difficult to solve analytically. One can represent the different combinations of information symbols b_j and elements $c_{i,j}$ using a decimal number representation that are converted into binary digits by (13). (See Appendix B)

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{m=0}^{2^{L \cdot U}-1} P(Z(m)) \cdot \lambda(\vec{r}|\mathcal{H}, Z(m), \vec{1})$$

$$e^{\sqrt{2\gamma}\sum_{i=0}^{L-1} Re\{r_i\}\sum_{j=0}^{U-1} z_{i,j}(m) - \gamma \sum_{i=0}^{U-1}\sum_{j=0}^{U-1}\langle\vec{z}_{*,i},\vec{z}_{*,j}\rangle - \beta \cdot TSC(Z(m))}$$

$$Z(m) = \{z_{i,j}\}_{L \times U}$$

$$z_{i,j} = c_{i,j} b_j$$

$$z_{i,j}(m) = (-1)^{1+mod\left(\left\lfloor\frac{m}{2^{j+i\cdot U}}\right\rfloor, 2\right)}$$

$$(13)$$

However, additional simplification comes from exploiting the invariance properties of the TSC and energy of the CDMA signal to rows, columns and sign inversion transformations of the spreading matrix. Using invariance transformations can be useful in simplifying generalized likelihood ratio. [8] The space of possibilities of the Z matrix can be partitioned in two subsets $\mathcal{S}_{\vec{a}}$ and $\mathcal{S}_{\vec{a},\tau}$ defined below.

Definition 1: The subset $\mathcal{S}_{\vec{a}}$ is the subset of matrices $Y = \{y_{i,j}\}^{L \times U}$ and $Z = \{z_{i,j}\}^{L \times U}$ that meet the following condition:

$$a_i = \left| \sum_{j=0}^{U-1} y_{i,j} \right| = \left| \sum_{j=0}^{U-1} z_{i,j} \right|$$

<div align="right">Eq. 14</div>

Definition 2: $\mathcal{S}_{\vec{a},\tau}$ is the subset of matrices $Y = \{y_{i,j}\}^{L \times U}$ and $Z = \{z_{i,j}\}^{L \times U}$ such that:

$$a_i = \left| \sum_{j=0}^{U-1} y_{i,j} \right| = \left| \sum_{j=0}^{U-1} z_{i,j} \right|$$

$$TSC(Y) = TSC(Z) = \tau$$

<div align="right">(15)</div>

The set $\mathcal{S}_{\vec{a},\tau}$ is a subset of $\mathcal{S}_{\vec{a}}$ by definition. The TST and energy are invariant under row/column permutations and row sign inversions.

The subsets $\mathcal{S}_{\vec{a}}$ are disjoint sets of all possible combinations of the matrix C. We can partition the each set $\mathcal{S}_{\vec{a}}$ in disjoint subsets $\mathcal{S}_{\vec{a},\tau}$ as illustrated in Figures 1 and 2.
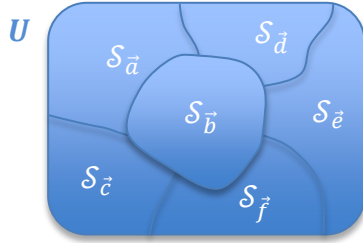
Figure 1. Partition Using Sets $\mathcal{S}_{\vec{a}}$
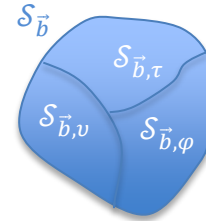
Figure 2. Partition Using Subset $\mathcal{S}_{\vec{b},\tau}$

Proposition 4: Let Z be a matrix subset $\mathcal{S}_{\vec{a},\tau}$, $\mathcal{P}_c$ a matrix permutation with binary elements {+1 or 0}, and $\mathcal{P}_s$ a diagonal matrix that consists of ±1 entries. Then the transformation:

$$Z' = \mathcal{P}_s \cdot Z \cdot \mathcal{P}_c$$

<div align="right">(16)</div>

is also contained in the subset $\mathcal{S}_{\vec{a},\tau}$. (See proof in Appendix C.)

We reformulate (11) in terms of summation terms using Definitions 1 and 2 in (14) and (15) as follows:

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{All\ \vec{a}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} \sum_{All\ \mathcal{P}_S} P(Z) \cdot \lambda(\vec{r}|\mathcal{H}, Z, \vec{1}).$$

$$(17)$$

It can be shown (see Appendices D and E) that the average likelihood becomes:

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{All\ \vec{a}} \alpha_{L,U,\vec{a}}(\beta) \prod_{i=0}^{L-1} e^{-\gamma \sum_{j=0}^{U-1} a_i^2} \cosh(\sqrt{2\gamma} \cdot \text{Re}\{r_i\} \cdot a_i)$$

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Count(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} e^{-\beta \cdot \tau}}{\sum_{All\ \vec{a}} \frac{2^L}{2^{Count(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} e^{-\beta \cdot \tau}}.$$

$$(18)$$

The likelihood function can be expressed as a product of cosh functions. The argument of each cosh function depends on the sum of the rows of matrix Z. The determination of all the subsets $\mathcal{S}_{\vec{a},\tau}$ introduces is extremely difficult because it requires fully characterizing $2^{L \cdot U}$ spreading matrices in terms of $\mathcal{S}_{\vec{a},\tau}$. There is no analytical formula available for such characterization; however, studying the behavior of $\alpha_{L,U,\vec{a}}(\beta)$ can lead us to find an alternative simplification.

3.4 Exact Solution of a 2x2 Spreading Matrix

The exact likelihood expression can be computed using Mathematica. A desktop computer can calculate the likelihood for code lengths $2 \leq L \leq 4$ as shown in Appendix F. Alternatively, we can compute $\alpha_{L,U,\vec{a}}(\beta)$ using the Matlab code provided in Appendix G.

The likelihood for a 2×2 spreading matrix hypothesis is given by:

$$\lambda(\vec{r}|\mathcal{H}) = \left(\frac{e^{-\langle \vec{r},\vec{r} \rangle}}{(\pi N_0)^2}\right)^{1/2} \left(\alpha_{2,2,[0,0]^T} \cosh(0 \cdot g_0) \cosh(0 \cdot g_1) + \right.$$

$$\alpha_{2,2,[2,0]^T} e^{-4\gamma} \cosh(2 \cdot g_0) \cosh(0 \cdot g_1) +$$

$$\alpha_{2,2,[0,2]^T} e^{-4\gamma} \cosh(0 \cdot g_0) \cosh(2 \cdot g_1) +$$

$$\left. \alpha_{2,2,[2,2]^T} e^{-8\gamma} \cosh(2 \cdot g_0) \cosh(2 \cdot g_1)\right)$$

where

$$\alpha_{2,2,[0,0]^T} = \alpha_{2,2,[2,2]^T} = \frac{1}{2}\frac{z}{1+z}\bigg|_{z=e^{-8\beta}}$$

$$\alpha_{2,2,[2,0]^T} = \alpha_{2,2,[0,2]^T} = \frac{1}{2}\frac{1}{1+z}\bigg|_{z=e^{-8\beta}}$$

$$g_i = \sqrt{2\gamma} \cdot Re\{r_i\}.$$

(19)

The behavior of $\alpha_{L,U,\vec{a}}(\beta)$ will be characterized in terms of infinite precision, i.e., $\lim_{\beta\to\infty}\alpha_{L,U,\vec{a}}(\beta)$. The convergence to zero provides an important simplification of (20). Several $\alpha_{L,U,\vec{a}}(\beta)$ parameters have been calculated for code lengths up to 4 as shown in Appendix H. The parameters that converge to a non-zero value have been identified with an asterisk in the Appendix.

Definition 3. The vector $\vec{a}^* = [a_0, \dots, a_{L-1}]$ is a feature vector if $C^* = \{c_{i,j}\}_{L\times U}$ is a spreading matrix that achieves the minimum TSC and:

$$a_i^\star = \left|\sum_{j=0}^{U-1} c_{i,j}^\star\right|$$

Property 5. The limit of $\alpha_{L,U,\vec{a}}(\beta)$ as β approaches infinity depends on the feature vectors as follows:

$$\lim_{\beta\to\infty}\alpha_{L,U,\vec{a}}(\beta) = \begin{cases} 0 & for\ a \neq \vec{a}^\star \\ \dfrac{\frac{2^L}{2^{Count(a_i=0)}}\left|\mathcal{S}_{\vec{a},\tau_{min}}\right|}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Count(a_i=0)}}\left|\mathcal{S}_{\vec{a},\tau_{min}}\right|} & for\ a = \vec{a}^\star \end{cases}$$

(20)

The proof is shown in Appendix J. For $\beta \to \infty$, the $\alpha_{L,U,\vec{v}}$ parameter is just the ratio of the cardinality of our previously defined sets $\mathcal{S}_{\vec{a},\tau}$ and $\mathcal{S}_{\vec{a}}$ multiplied by a constant. Increasing the precision is analogous to deriving a K-Means classifier from the Gaussian Mixture Models. [8] Figures 3 and 4 show plots of $\alpha_{3,2,\vec{a}}(\beta)$ in logarithmic scales. It can be appreciated that $\alpha_{3,2,\vec{a}}(\beta)$ converges very rapidly for $\beta \geq 1$. Similar observations were made for code length cases $2 \leq L \leq 6$. (See Appendix I) For code length of higher length, we would have to make the assumption that these observations are also valid; otherwise, the $\alpha_{L,U,\vec{a}}(\beta)$ coefficients would need to be calculated. This is already a computationally intractable problem.

For antipodal matrices, we provide the following empirical rule for computing the parameters $\alpha_{L,U,\vec{a}}$. First, we find any matrix $C^{\star}$ with minimum TSC. Then we multiply the matrix by all possible 2^U choices of the vector $\vec{b}$ and take the absolute value of each element of the resulting vector. The result is a set of feature vectors $\vec{a}^{\star}$. The limiting value of $\alpha_{L,U,\vec{a}}$ is the proportion of a given vector $\vec{a}$ in the set of 2^U possibilities. (See Appendix J)

$$\lim_{\beta \to \infty} \alpha_{L,U,\vec{a}^{\star}}(\beta) = \frac{count(abs(C^{\star}\vec{b}) = \vec{a}^{\star})}{2^U}, \qquad for\ all\ possible\ \vec{b}.$$

(21)

$\alpha_{3,3,0}[\beta]$

Log[$\alpha_{3,3,0}[\beta]$]

$$\frac{3\ (2 + 6\ z + z^3)}{4\ (6 + 9\ z + z^3)}$$

$\alpha_{3,3,1}[\beta]$

Log[$\alpha_{3,3,1}[\beta]$]

$$\frac{3\ (2 + z)}{4\ (6 + 9\ z + z^3)}$$
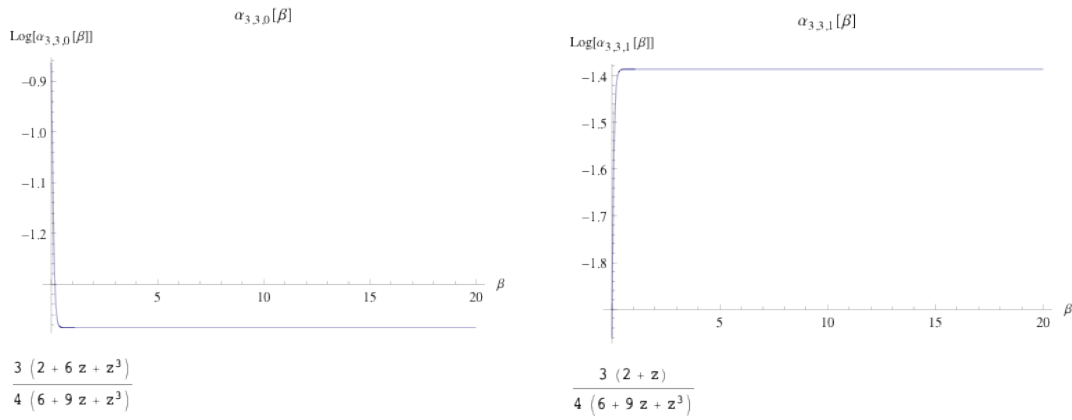
Figure 3. Non Vanishing Coefficients for 3x3 Spreading Matrix

$\alpha_{3,3,2}[\beta]$

Log[$\alpha_{3,3,2}[\beta]$]

$$\frac{3\ z}{4\ (6 + 9\ z + z^3)}$$

$\alpha_{3,3,3}[\beta]$

Log[$\alpha_{3,3,3}[\beta]$]

$$\frac{z^3}{4\ (6 + 9\ z + z^3)}$$
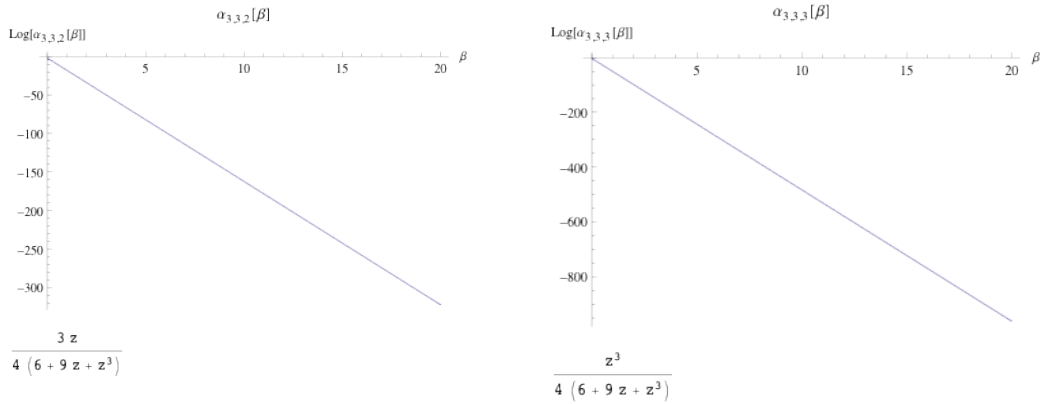
Figure 4. Vanishing Coefficients for 3x3 Spreading Matrix

In order to prove this assertion (21), we need to demonstrate that two antipodal matrices C_1 and C_2 with same minimum TST and different sum of rows are always related by rows and column permutations $C_1 = \mathcal{P}_1 C_2 \mathcal{P}_2$. This property is referred

as a Hadamard equivalence, but unfortunately it does not always hold. [10] Therefore, it would be necessary to find all possible $C^\star$ that are not equivalent or constrain our set to a specific construction. In our case, we will constrain our problem to the Sylvester's Construction of the Hadamard matrix. The coding theory does not provide yet an answer for finding an expression of the lowest TSC matrices of an arbitrary size.

3.5 Average Likelihood Function Example

In this section we are going to summarize the steps needed for constructing the likelihood function given the hypothesis $\mathcal{H} = \{4,4\}$.

Step 1: Find a matrix with minimum TSC

This step is easy for Hadamard matrices with $L = U = 2^k$.

$$C^\star = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Step 2: Generate the feature vectors using process described in [Eq. 21]:

$$\vec{a}_1^\star = [4,0,0,0]^T, \alpha_{L,U,\vec{a}_1^\star} = 2/2^4 = 1/8$$

$$\vec{a}_2^\star = [2,2,2,2]^T, \alpha_{L,U,\vec{a}_2^\star} = 8/2^4 = 1/2$$

Step 3: Form the product of hyperbolic cosine functions for the different permutations of the feature vectors.

Permutations of $\vec{a}_1^\star$:

$$[4,0,0,0]^T, [0,4,0,0]^T, [0,0,4,0]^T, [0,0,0,4]^T$$

For each one $\alpha_{L,U,\vec{a}^\star} = 2/2^4 = 1/8$

Permutations of $\vec{a}_2^\star$:

$$[2,2,2,2]^T$$

with $\alpha_{L,U,\vec{a}^\star} = 8/2^4 = 1/2$

Product of cosh functions:

$$\lambda(\vec{r}|\mathcal{H} = \{4,4\}) = \left(\frac{e^{-\langle \vec{r},\vec{r} \rangle}}{(\pi N_0)^2}\right)^{1/2} \left(\frac{1}{8}\sum_{i=0}^{3} e^{-4^2\gamma}\cosh(4 \cdot \sqrt{2\gamma} \cdot Re\{r_i\}) + \right.$$

$$\left. \frac{1}{2}\prod_{i=0}^{3} e^{-2^2\gamma}\cosh(2 \cdot \sqrt{2\gamma} \cdot Re\{r_i\}) \right)$$

As the code length and number of user grow, the average likelihood becomes complicated, but the complexity is much less than the original formulation found in (11). The 2^{20} unknown variables have been totally removed from the original expression. In the next section, the expression will be simplified by adding a constraint to the problem.

3.6 ROC Using Simplified Likelihood

Figures 5 through 14 show ROC curves for several simple cases of spreading codes. The length of the signal under test is expressed as the total number of chip. This value is based on selecting $5! \cdot 2 = 240$ symbols of the hypothesis with the highest code length. If the code length is 2 (see Figure 5), length of the signal is 480 chips. The ROC curves are plotted for different SNR values. All of the simulations correspond to $\beta \to \infty$. Cases $\mathcal{H} = \{L, 1\}$ are trivial ones. Their likelihood ratio reduces to a constant when compared to a BPSK signal and it is impossible to classify them correctly as shown in Figure 6.

Figure 5. ROC curves for ALRT Test $\mathcal{H}_1 = \{2, 2\}$ vs. $\mathcal{H}_0 = \{1, 1\}$

Figure 6. ROC curves for ALRT Test $\mathcal{H}_1 = \{2, 1\}$ vs. $\mathcal{H}_0 = \{1, 1\}$

Figure 7. ROC curves for ALRT Test $\mathcal{H}_1 = \{3, 3\}$ vs. $\mathcal{H}_0 = \{1, 1\}$

Figure 8. ROC curves for ALRT Test $\mathcal{H}_1 = \{3, 3\}$ vs. $\mathcal{H}_0 = \{2, 2\}$

Figure 9. ROC curves for ALRT Test
$\mathcal{H}_1 = \{3, 3\}$ vs. $\mathcal{H}_0 = \{3, 2\}$
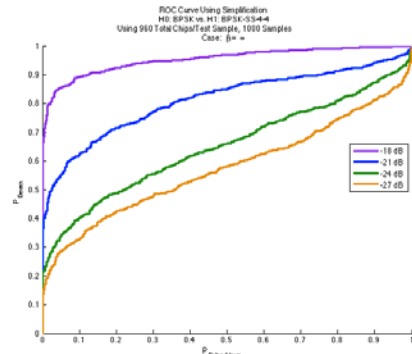
Figure 10. ROC curves for ALRT Test
$\mathcal{H}_1 = \{3, 3\}$ vs. $\mathcal{H}_0 = \{2, 2\}$
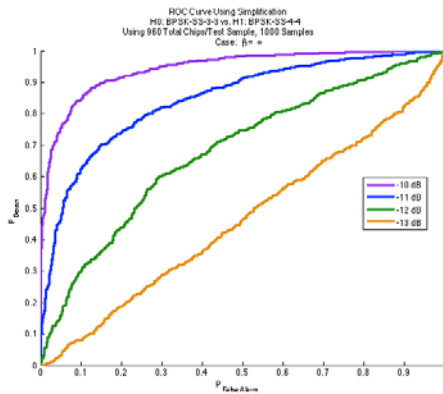
Figure 11. ROC curves for ALRT Test
$\mathcal{H}_1 = \{4, 4\}$ vs. $\mathcal{H}_0 = \{3, 3\}$
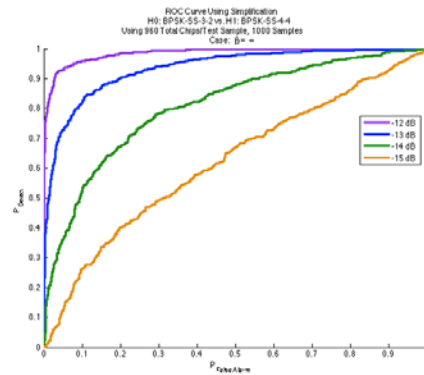
Figure 12. ROC curves for ALRT Test
$\mathcal{H}_1 = \{4, 4\}$ vs. $\mathcal{H}_0 = \{3, 2\}$
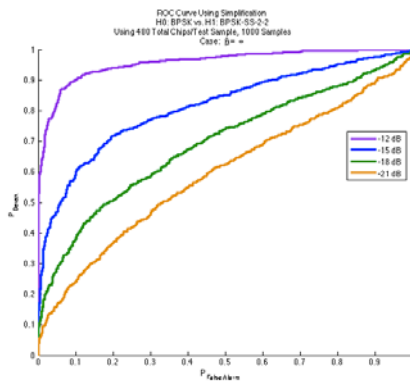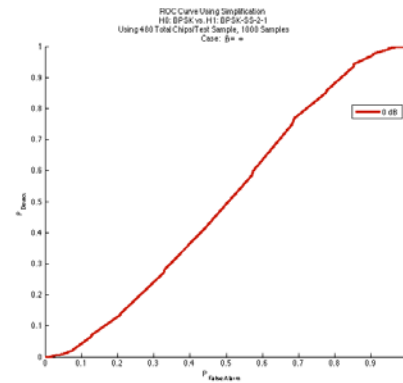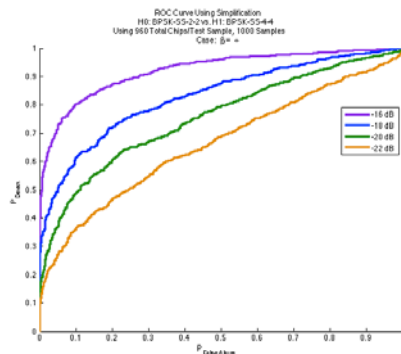

Figure 13. ROC curves for ALRT Test
$\mathcal{H}_1 = \{4, 4\}$ vs. $\mathcal{H}_0 = \{2, 2\}$
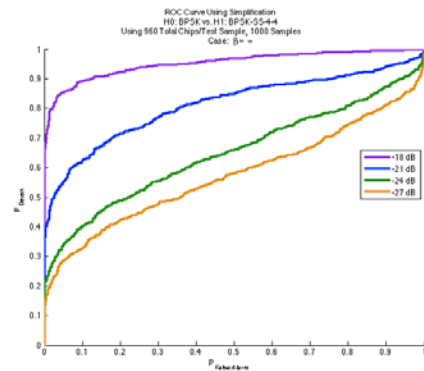
Figure 14. ROC curves for ALRT Test
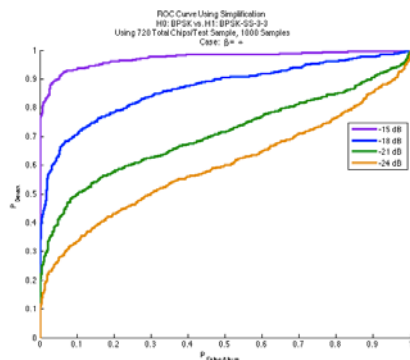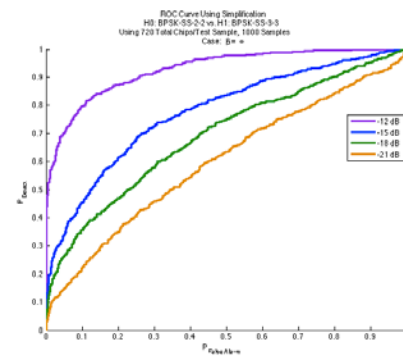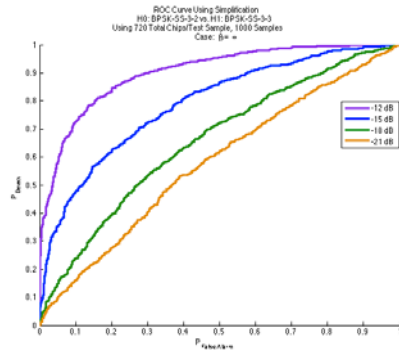$\mathcal{H}_1 = \{4, 4\}$ vs. $\mathcal{H}_0 = \{1, 1\}$

The orange curves are extreme cases where the chip SNR is so low that the classifier starts breaking. As more noise is added to the signals, the ROC curves invert their

performance. A noisy CDMA signal under hypothesis null hypothesis $\mathcal{H}_0$ will look like $\mathcal{H}_1$. This might be a result of numerical precision errors.

The developed classifier is a hard decision scheme. We have defined the joint probability of the spreading code in terms of $P(C)$, but we can also define the probability in terms of the TSC as $P(\tau|C;\beta)$. As it is expected from hard decision classifiers, the probabilities becomes discrete instead of continuous: $P(\tau = \tau_{min}|C;\beta) \approx 1$ and $P(\tau \neq \tau_{min}|C;\beta) \approx 0$.

We might expect that codes with a slightly deviation from the minimum TSC spreading matrices would be detected and classified correctly. The main parameters affecting the performance of the classifier are the parameter β and the feature vector $\vec{a}^\star$. If we refer to Figure 4, we can notice that for a small precision such as $\beta \geq 1$, the coefficients $\alpha_{L,U,\vec{a}}(\beta)$ are already small enough to consider valid the approximation of Equation (20). So, the parameter that mostly affects the performance of the classifier must be the parameter vector. If two codes has similar parameter vector, then the classifier rule should also work. This is only an assessment. Future work must consider the scenario of small deviations from the minimum TSC codes.

3.7 Classification of CDMA Signals based on Sylvester's Construction

We consider the simplified hypothesis $\mathcal{H} = \{L, L\}$ using the Sylvester's Construction of the Hadamard matrix.

Definition 4: A Hadamard matrix H is a $L \times L$ matrix of elements ± 1 that satisfies the following condition [11]:

$$H^T H = L \cdot I_L$$

$$(22)$$

I_L is an $L \times L$ identity matrix.

Definition 5: The Sylvester's Construction of the Hadamard matrix is given by the following recursive formula using the Kronecker product [10]:

$$H_2 = \begin{bmatrix} +1 & +1 \\ +1 & -1 \end{bmatrix}$$

$$H_{2^k} = H_2 \otimes H_{2^{k-1}}$$

$$(23)$$

Hadamard matrices have been a subject of study of mathematicians for years. There exists several constructions known at the present time; however, there is no theorem that proves the existence for any arbitrary code length [8].

It is possible to generate distinct Hadamard matrices using the Kronecker product of two Hadamard matrices.

Theorem 2: The Total Squared Correlation of a matrix generated from a Kronecker product is given by: [12]

$$TSC(H_a \otimes H_b) = TSC(H_a)TSC(H_b) + L_a U_a TSC(H_b) + L_b U_b TSC(H_a)$$

(24)

Corollary 2-1: The Kronecker product of two Hadamard matrices is another Hadamard matrix.

For a Hadamard matrix the TSC is zero based on Equation (3). The expression in (24) is zero under the Kronecker product of two Hadamard matrices.

Corollary 2-2: The Total Square Correlation of a Hadamard matrix is zero.

Unfortunately the Kronecker product of two minimum TSC matrices does not necessarily produce a matrix with minimum TSC unless both matrices are Hadamard.

The feature vector corresponding to a Sylvester's Construction is given by:

$$\vec{\xi}^\star = [\xi_0 = L, \xi_1 = 0, \dots, \xi_{L-1} = 0]^T$$

(25)

as well as the L possible row permutations of the feature vector $\vec{\xi}^\star$. This is a fact known from the Sylvester's Construction Hadamard Matrix.

By restricting our set $\mathcal{S}_{\vec{a}^\star,0}$ to $\mathcal{S}_{\vec{\xi}^\star,0}$ we eliminate other constructions of Hadamard matrices by having setting our ratio to be one:

$$\lim_{\beta \to \infty} \alpha_{L,U,\vec{\xi}^\star}(\beta) = \frac{\frac{2^L}{2^{Count(\xi_i=0)}} \left| \mathcal{S}_{\vec{\xi}^\star,0} \right|}{\frac{2^L}{2^{Count(\xi_i=0)}} \left| \mathcal{S}_{\vec{\xi}^\star,0} \right|} = 1$$

(26)

The average likelihood function of such scheme reduces to the following expression:

$$\lambda(\vec{r}|\mathcal{H} = H_L) = \left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \sum_{\substack{All \\ Permutations \\ of\ \vec{\xi}^\star\ elements}} \prod_{i=0}^{L-1} e^{-\gamma|\xi_i|^2} \cosh\left(\sqrt{2\gamma} \cdot \xi_i \cdot Re\{r_i\}\right).$$

(27)

For the case $L = 1$, the expression reduces to the average likelihood function of a BPSK signal found in [1].

The construction of the likelihood ratio between hypotheses $\mathcal{H}_{2L} = H_{2L}$ and $\mathcal{H}_L = H_L$ provides additional simplification. (See Appendix K) For every vector $\vec{r}_{2L\times1}$ in $\mathcal{H}_{2L}$ there are two vectors $\vec{r}_{L\times1}$ in $\mathcal{H}_L$. The two vectors are statistically independent, so the likelihood in the denominator is multiplied. The terms $e^{-\langle\vec{r},\vec{r}\rangle}$ containing the energy of the vectors $\vec{r}_{2L\times1}$ and $\vec{r}_{L\times1}$ are cancelled. The final form is given by:

$$\frac{\lambda(\vec{r}|\mathcal{H}_{2L})}{\lambda(\vec{r}|\mathcal{H}_L)} = \frac{\sum_{k=0}^{2L} \cosh(\sqrt{2\gamma} \cdot 2L \cdot Re\{r_k\})}{\sum_{k=0}^{L} \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\}) \cdot \sum_{k=L+1}^{2L} \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\})}$$

(28)

Using the double angle formulas for cosh functions helps reducing the amount of calculations by reusing the terms in the denominator.

$$\Lambda_{2L}(\vec{r}) = \frac{\lambda(\vec{r}|\mathcal{H}_{2L})}{\lambda(\vec{r}|\mathcal{H}_L)} = \frac{\sum_{k=0}^{2L}(2d_k^2 - 1)}{\sum_{k=0}^{L} d_k \cdot \sum_{k=L+1}^{2L} d_k}$$

$$d_k = \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\})$$

(29)

The product of cosh functions can cause numerical overflow in the computations. The likelihood ratio shown in the previous equation offers better numerical stability. Nevertheless, this simplification implies that full Hadamard CDMA signals have easily implementable rules and can be easily detected.

4. Results

The ratio test in (28) allows computing the likelihood ratio for matrices of higher code length with low complexity. The Figures 15 through 26 show the likelihood ratio calculated for two sets. The length of the signal under test is expressed as the total number of chip. This value is based on selecting $5! \cdot 2 = 240$ symbols of the

hypothesis with the highest code length. If the code length is 2 (see Figure 15), length of the signal is 480 chips.
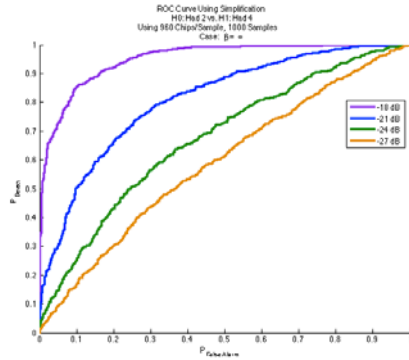
Figure 15. ROC Curves Hadamard 4 versus 2
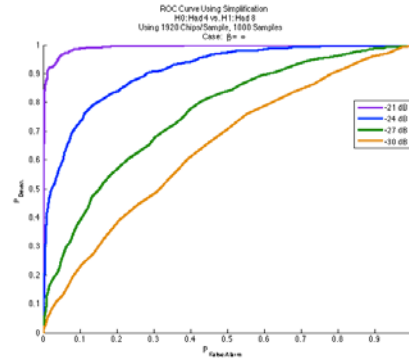
Figure 16. ROC Curves Hadamard 8 versus 4
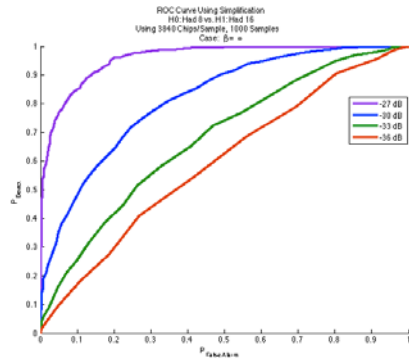
Figure 17. ROC Curves Hadamard 16 versus 8
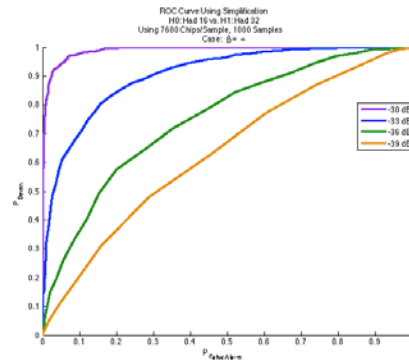
Figure 18. ROC Curves Hadamard 32 versus 16
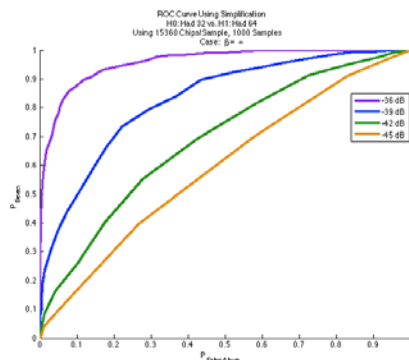
Figure 19. ROC Curves Hadamard 64 versus 32
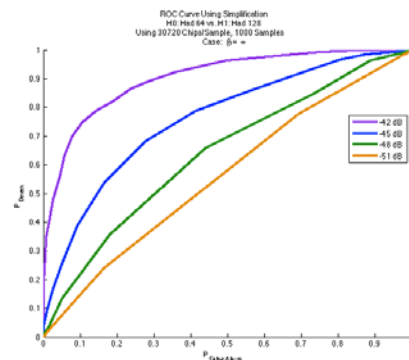
Figure 20. ROC Curves Hadamard 128 versus 64
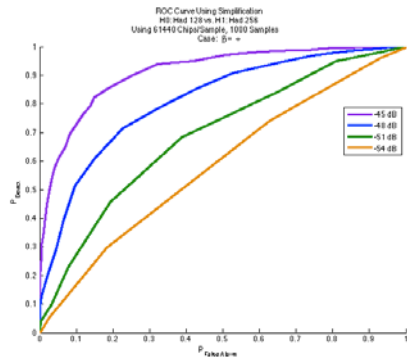
Figure 21. ROC Curves
Hadamard 256 versus 128
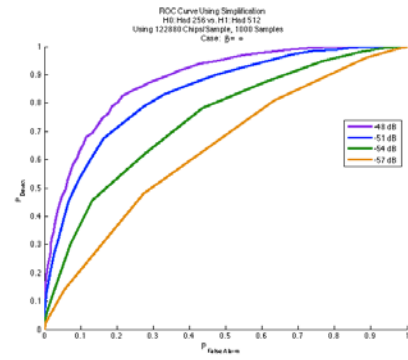
Figure 22. ROC Curves
Hadamard 512 versus 256
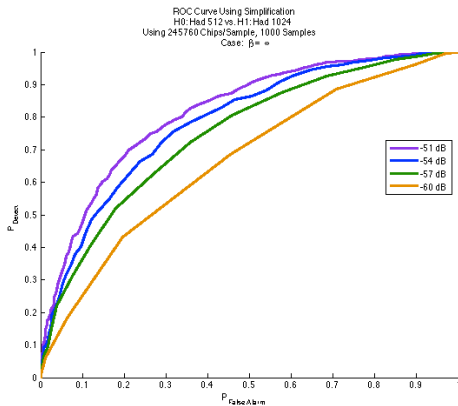
Figure 23. ROC Curves
Hadamard 1024 versus 512
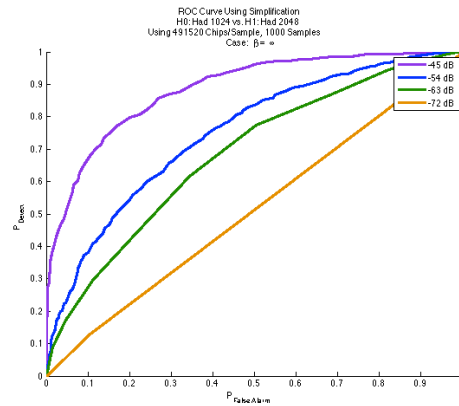
Figure 24. ROC Curves
Hadamard 2048 versus 1024
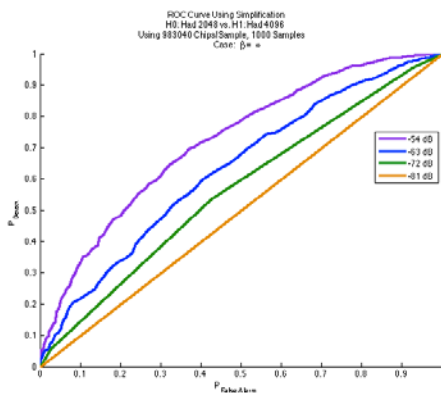
Figure 25. ROC Curves
Hadamard 4096 versus 2048
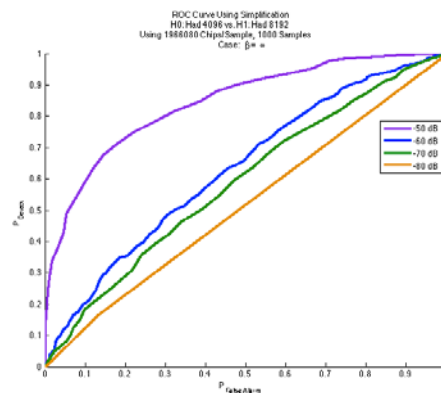
Figure 26. ROC Curves
Hadamard 8192 versus 4096

The chip-level SNR appears to be extremely low. First, we note that the chip-level SNR is a quantity defined for convenience and can be expressed in terms of a more

meaningful quantity: the symbol-level SNR. The symbol-level SNR is the energy to noise power density ratio of one symbol:

$$\gamma_s = \frac{E}{N_0} = \gamma \cdot L.$$

$$dB(\gamma_s) = dB(\gamma_c) + 3.01 \cdot k.$$

(30)

The symbol SNR equals the chip SNR plus 3 dB times the code length L of the Hadamard matrix. The symbol SNR should not be confused with the total SNR of the CDMA signal.

The low SNR can be explained by understanding the process of generating a CDMA signal. When a full load CDMA signal is constructed, there is a probability p_{spike} for finding one single spike of magnitude L and energy L^2 within a frame of $L - 1$ zeros. The signal classifier can be interpreted as a detector of spikes with probability p_{spike} equals to Equation (21).

5. Conclusions

1. This research demonstrated that is possible to construct a classifier for the detection of full-loaded CDMA signals.

2. The simplest CDMA classifier is a hard decision classifier that only considers the matrix that achieves the lowest Total Squared Correlation for a given code length.

3. Further simplification of the likelihood equations can be made if a particular construction of the spreading code is assumed. Such is the case of Sylvester's construction of Hadamard matrices.

This approach defines the probability of a low correlation matrix and uses this definition for averaging over all possible spreading matrices. The resulting likelihood is a sum of the product of two terms as shown in (18). The first term depends on the Total Square Correlation and a parameter β and the second term depends on the feature vectors $\vec{a}^\star$. The first term acts like a filter that dampers the effect of many of the products of cosh functions. After dropping the insignificant terms, the expression terms depend exclusively on the spreading matrix that achieves the minimum Total Square Correlation.

The detection of Hadamard matrices generated using the Sylvester's Construction adds more simplification to the likelihood function. The sum of the product of cosh functions reduces to only one summation as shown in (29). Although the expression is in a simple form, numerical overflow is a concern for any implementation of the likelihood function using large code lengths.

6. Future Work

The case of balanced CDMA signal, i.e., equal energy signals provides a limited application of the theory. The next step in this research will be the assumption of different energy levels for each signature vector. Also, the case of QPSK CDMA will be investigated. Preliminary research has shown that it is possible to develop similar equations in terms of the product of cosh functions and feature vectors.

7. Bibliography

[1]   C. Huang and A. Polydoros, "Likelihod Methods for MPSK Modulation Classification," *IEEE Transactions on Communications,* 1995.

[2]   Y. Yao and V. Poor, "Eavesdropping in the Synchronous CDMA Channel: An EM-Based Approach," in *IEEE Transactions on Signal Processing*, 2001.

[3]   Y. Yao and V. Poor, "EM-Based Blind Demodulation of Synchronous CDMA," *IEEE 6th Sym. On Spread Spectrum Tech. and Applications,* 2000.

[4]   O. Dobre, A. Abdi and W. Su, "Survey of Automatic Modulation Classification Techniques: Classical Approaches and New Trends," *IET Communications,* 2007.

[5]   J. Xu, W. Su and M. Zhou, "Likelihood Ratio Approaches to Automatic Modulation Classification," 2011.

[6]   S. Khamy, A. Helw and A. Mahdy, "Classification of Multirate CDMA Signals using Compressed Cyclostationary Features," *Progress in Electromagnetics Research Proceedings,* 2012.

[7]   P. Cotae, "Distributive Iterative Method for Minimizing Generalized Total Square Correlation of CDMA Systems," *IEEE ITW,* 2003.

[8]   C. Bishop, Patter Recognition and Machine Learning, Springer, 2006.

[9]   P. Schreier, L. Scharf and A. Hanssen, "A Generalized Likelihood Ratio Test for Impropriety of Complex Signals," in *IEEE Signal Processing Letters*, 2006.

[10] "Encyclopeadia of Design Theory," School of Mathematics Sciences, Queen Mary, University of London, [Online]. Available: designtheory.org/library/encyc. [Accessed 16 September 2013].

[11] E. Assmus and J. Key, "Hadamrad Matrices and their Designs," *Transactions of the American Mathematical Society,* vol. 330, no. 1, 1992.

[12] P. Lancaster and M. Tesmenetsky, Theory of Matrices with Applications, Academic Press, 1985.

[13] J. Hamkins, "Low Complexity Approximation to Maximum Likelihood MPSK Modulation Classification," in *Jet Propulsion Laboratory, National Aeronautics and Space Administration*, 2004.

[14] F. Hameed, O. Dobre and D. Popescu, "On Likelihood-Based Approach to Modulation Classification," *IEEE Transactions on Wireless Communications,* vol. 8, no. 12, 2009.

[15] A. Vega Irizarry, "Optimal Methods for Classification of Digitally Modulated Signals," AFRL/DTIC, 2013.

APPENDIX A.   Likelihood Form

We define the following variables:

Signal to noise ratio per chip	$$\gamma_c = \frac{E}{N_0 L},$$
Unnormalized signal	$$\vec{s} = C \cdot \vec{b}$$
CDMA transmit signal	$$\vec{x} = \sqrt{E/L}\,\vec{s}$$
Output of the correlator	$$\vec{r} = \frac{\vec{y}}{\sqrt{N_0/2}}.$$

Then, we proceed to substitute them in the likelihood function:

$$\lambda\big(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\big) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{(\vec{y}-\vec{x})^H \Sigma^{-1}(\vec{y}-\vec{x})}{2}}$$

$$\lambda\big(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\big) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\left(\sqrt{\frac{N_0}{2}}\vec{r}-\sqrt{E/L}\,C\vec{b}\right)^H \left(\sqrt{\frac{N_0}{2}}\vec{r}-\sqrt{E/L}\,C\vec{b}\right)}{2N_0/2}}$$

$$\lambda\big(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\big) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\frac{N_0}{2}\vec{r}^H\vec{r}+2\sqrt{\frac{E}{L}}Re\{\sqrt{N_0/2}\,\vec{r}^H C\vec{b}\}-\frac{E}{L}(C\gamma\vec{b})^H C\vec{b}}{N_0}}$$

$$\lambda\big(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\big) = \frac{1}{(\pi N_0)^{L/2}} e^{-\frac{\vec{r}^H\vec{r}}{2}+\sqrt{2\gamma}Re\{\vec{r}^H\}C\vec{b}-\gamma(C\vec{b})^H C\vec{b}}$$

The final result is:

$$\lambda\big(\vec{y}|\mathcal{H} = \{L, U\}, C, \vec{b}\big) = \left(\frac{e^{-(\vec{r},\vec{r})}}{(\pi N_0)^L}\right)^{1/2} e^{\sqrt{2\gamma}Re\{\vec{r}^H\}C\vec{b}-\gamma(C\vec{b})^H C\vec{b}}.$$

APPENDIX B.   Average Likelihood Function

Consider the base 2 and decimal representation of $\vec{b}$ and C combinations as follows:

$$n_{10} = (B_{U-1}, \dots, B_1, B_0)_2$$

$$m_{10} = (C_{L \cdot U - 1}, \dots, C_1, C_0)_2.$$

The multiple summations over binary symbols b_j and $c_{i,j}$ can be implemented with a summation of the decimal indexes n and m respectively:

$$b_j(n) = (-1)^{1 + mod\left(\left\lfloor \frac{n}{2^j} \right\rfloor, 2\right)}$$

$$c_{i,j}(m) = (-1)^{1 + mod\left(\left\lfloor \frac{m}{2^{j + U \cdot i}} \right\rfloor, 2\right)}$$

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{m=0}^{2^{L \cdot U} - 1} \frac{1}{2^U} \sum_{n=0}^{2^U - 1} P\big(C(m)\big) \cdot \lambda(\vec{r}|\mathcal{H}, C(m), b(n))$$

A new variable $z_{i,j}$ equivalent to the product b_j and $c_{i,j}$ eliminates the need of one summation.

$$z_{i,j}(p) = c_{i,j} \cdot b_j = (-1)^{1 + mod\left(\left\lfloor \frac{p}{2^{j + i \cdot U}} \right\rfloor, 2\right)}$$

The likelihood reduces to:

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{M=0}^{2^{L \cdot U} - 1} P\big(Z(m)\big) \cdot \lambda\big(\vec{r}|\mathcal{H}, Z(m), \vec{1}\big)$$

where:

$$\lambda\big(\vec{r}|\mathcal{H}, Z(m), \vec{1}\big) = e^{\sqrt{2\gamma} \sum_{i=0}^{L-1} r_i \sum_{j=0}^{U-1} z_{i,j}(m) - \gamma \sum_{j=0}^{U-1} \langle \vec{z}_{*,j}(m), \vec{z}_{*,j}(m) \rangle - \beta \cdot \tau(Z(m))}$$

$$Z(m) = \{z_{i,j}\}_{L \times U} = [\vec{z}_{*,0}, \vec{z}_{*,1}, \dots, \vec{z}_{*,U-1}]$$

and

$$\vec{1} = [1, \dots, 1]^T.$$

Note that the metric is invariant to this transformation because $|b_i|^2 = 1$.

$$\tau(C) \triangleq \sum_{i=0}^{U-1} \sum_{\substack{j=0 \\ j \neq i}}^{U-1} \left| \langle \vec{c}_{*,i}, \vec{c}_{*,j} \rangle \right|^2$$

$$\tau(Z, \vec{b}) = \sum_{i=0}^{U-1} \sum_{\substack{j=0 \\ j \neq i}}^{U-1} \left| \frac{1}{b_i} \frac{1}{b_j} \langle \vec{z}_{*,i}, \vec{z}_{*,j} \rangle \right|^2$$

$$\tau(Z) = \sum_{i=0}^{U-1} \sum_{\substack{j=0 \\ j \neq i}}^{U-1} \left| \langle \vec{z}_{*,i}, \vec{z}_{*,j} \rangle \right|^2 = \tau(C)$$

Definition 1: The subset $\mathcal{S}_{\vec{a}}$ is the subset of matrices $Y = \{y_{i,j}\}^{L \times U}$ and $Z = \{z_{i,j}\}^{L \times U}$ that meet the following condition:

$$a_i = \left| \sum_{j=0}^{U-1} y_{i,j} \right| = \left| \sum_{j=0}^{U-1} z_{i,j} \right|$$

Eq. 31

Definition 2: $\mathcal{S}_{\vec{a},\tau}$ is the subset of matrices $Y = \{y_{i,j}\}^{L \times U}$ and $Z = \{z_{i,j}\}^{L \times U}$ such that:

$$a_i = \left| \sum_{j=0}^{U-1} y_{i,j} \right| = \left| \sum_{j=0}^{U-1} z_{i,j} \right|$$

$$TSC(Y) = TSC(Z) = \tau$$

Eq. 32

Proposition 1: The TSC is invariant to permutations and sign-inversions of rows and columns.

Proof: A permutation matrix $\mathcal{P}$ has the following property:

$$\mathcal{P} \cdot \mathcal{P}^T = I$$

Using our definition of a TSC for a binary antipodal matrix [Eq. 4],

$$TSC(\mathcal{P}_1 C \mathcal{P}_2) = \|\mathcal{P}_2^T C^T \mathcal{P}_1^T \mathcal{P}_1 C \mathcal{P}_2\|_F^2 - L \cdot U$$

$$= \|C^T \mathcal{P}_1^T \mathcal{P}_1 C \mathcal{P}_2 \mathcal{P}_2^T\|_F^2 - L \cdot U$$

$$= \|C^T C\|_F^2 - L \cdot U$$

$$= TSC(C)$$

Proposition 2: The TSC is invariant to transposition.

Proof: The Frobenius norm is invariant to transposition. The term $L \cdot U$ does not change after transposing the matrix C.

As an important note, it is interesting to mention that Hadamard matrices are known to be invariant to permutations of rows and columns, sign inversions of rows and columns and transposition. [10]

Proposition 3: The energy of a signal is invariant under the following transformation using permutation matrices:

$$\vec{b}' = \mathcal{P}_1\vec{b}$$

$$C' = \mathcal{P}_2 C \mathcal{P}_1^T$$

Proof: This is an easy proof using permutation matrices. The energy of a CDMA per symbol signal is given by:

$$\vec{x}^T\vec{x} = \gamma \vec{b}^T C^T C \vec{b}$$

$$\vec{x}'^T\vec{x}' = \gamma \vec{b}'^T C'^T C' \vec{b}'$$

$$\vec{x}'^T\vec{x}' = \gamma \left(\mathcal{P}_1\vec{b}\right)^T (\mathcal{P}_2 C \mathcal{P}_1^T)^T (\mathcal{P}_2 C \mathcal{P}_1^T)\left(\mathcal{P}_1\vec{b}\right)$$

$$\vec{x}'^T\vec{x}' = \gamma \vec{b}'^T \mathcal{P}_1^T \mathcal{P}_1 C^T \mathcal{P}_2^T \mathcal{P}_2 C \mathcal{P}_1^T \mathcal{P}_1 \vec{b}$$

$$\vec{x}'^T\vec{x}' = \gamma \vec{b}^T C^T C \vec{b}$$

$$\vec{x}'^T\vec{x}' = \vec{x}^T\vec{x}$$

Proposition 4: Let Z be a matrix subset $\mathcal{S}_{\vec{a},\tau}$, $\mathcal{P}_c$ a matrix permutation with binary elements {+1 or 0}, and $\mathcal{P}_s$ a diagonal matrix that consists of ±1 entries. Then the transformation:

$$Z' = \mathcal{P}_s \cdot Z \cdot \mathcal{P}_c$$

Eq. 33

is also contained in the subset $\mathcal{S}_{\vec{a},\tau}$.

Proof: It is assumption that $Z \in \mathcal{S}_{\vec{a},\tau}$.with:

$$a_i = \left|\sum_{j=0}^{U-1} z_{i,j}\right|, and$$

$$TSC(Y) = TSC(Z) = \tau.$$

Also, it is claimed that $Z' \in \mathcal{S}_{\vec{a},\tau}$. under $Z' = \mathcal{P}_s \cdot Z \cdot \mathcal{P}_c$. Therefore,

$$a_i = \left| \sum_{j=0}^{U-1} z'_{i,j} \right| = \left| \mathcal{P}_s \cdot Z \cdot \mathcal{P}_c \cdot \vec{1} \right| = \left| (\mathcal{P}_s)_{i,i} \sum_{k=0}^{U-1} \sum_{j=0}^{U-1} z_{i,j} (\mathcal{P}_c)_{j,k} \cdot 1 \right|$$

$$a_i = \left| \sum_{j=0}^{U-1} z'_{i,j} \right| = \left| \sum_{j=0}^{U-1} z_{i,j} \right|$$

The expression $(\mathcal{P}_c)_{j,k}$ does not change the sign of the column $z_{*,j}$. It only sorts the columns. Also,

$$TSC(Z) = TSC(Z') = \tau$$

by Proposition 1. Therefore, under the transformation $Z' = \mathcal{P}_s \cdot Z \cdot \mathcal{P}_c$, the transformation is also part of the subset: $Z' \in \mathcal{S}_{\vec{a},\tau}$.

APPENDIX D.   Simplification of the Likelihood Function

The next step in the simplification of the likelihood requires partitioning the matrix space Z in subsets of equal TSC and energy. Averaging over the likelihood will be replaced with an average over matrices belonging to all subsets $\mathcal{S}_{\vec{a}}$ and all the possible sign inversions.

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{All\ \vec{a}} \frac{1}{2^{Sum(a_i=0)}} \sum_{Z' \in \mathcal{S}_{\vec{a}}} \sum_{All\ \mathcal{P}_s} P(Z') \cdot \lambda(\vec{r}|\mathcal{H}, Z', \vec{1})$$

An element of $\vec{a}$ that equals zero introduces double counting of spreading matrices in the subset $\mathcal{S}_{\vec{a}}$. To correct this, we must divide by 2 raised to the sum of all zero elements in $\vec{a}$.

The entire summation can be calculated if the whole set $\mathcal{S}_{\vec{a}}$ is known for every possible vector $\vec{a}$. The summation over the sign inversion matrices $\mathcal{P}_s$ can be simplified as follows:

$$\sum_{All\ \mathcal{P}_s} P(Z) \cdot \lambda(\vec{r}|\mathcal{H}, Z, \vec{1}) = e^{-\gamma \sum_{i=0}^{U-1} \langle \vec{z}_{*,i}, \vec{z}_{*,i} \rangle - \beta \cdot \tau(Z)} \sum_{All\ \rho_{i,i}=\pm 1} e^{\sqrt{2\gamma} \sum_{i=0}^{L-1} \mathrm{Re}\{r_i\} \rho_{i,i} \sum_{j=0}^{U-1} z_{i,j}}$$

Using Theorem 1 of Appendix E, results in

$$\sum_{All\ \rho_{i,i}=\pm 1} e^{\sqrt{2\gamma} \sum_{i=0}^{L-1} \mathrm{Re}\{r_i\} \rho_{i,i} \sum_{j=0}^{U-1} z_{i,j}} = \sum_{\rho_{0,0}=\pm 1} \cdots \sum_{\rho_{L-1,L-1}=\pm 1} e^{\sum_{i=0}^{L-1} \rho_{i,i}\left(\sqrt{2\gamma}\mathrm{Re}\{r_i\}\sum_{j=0}^{U-1} z_{i,j}\right)}$$

$$= 2^L \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma} \cdot r_i \sum_{j=0}^{U-1} z_{i,j}\right)$$

$$= 2^L \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma} \cdot r_i \left|\sum_{j=0}^{U-1} z_{i,j}\right|\right)$$

$$= 2^L \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma} \cdot r_i \cdot a_i\right)$$

So,

$$\sum_{All\ \mathcal{P}_s} P(Z) \cdot \lambda(\vec{r}|\mathcal{H}, Z, \vec{1}) = 2^L e^{-\beta \cdot \tau(Z) - \gamma \sum_{j=0}^{U-1}|a_i|^2} \prod_{i=0}^{L-1} \cosh\left(\sqrt{2\gamma} \cdot \mathrm{Re}\{r_i\} \cdot a_i\right)$$

$$= 2^L e^{-\beta \cdot \tau(Z)} \prod_{i=0}^{L-1} e^{-\gamma |a_i|^2} \cosh(\sqrt{2\gamma} \cdot \text{Re}\{r_i\} \cdot a_i)$$

The sum over $\mathcal{S}_{\vec{a}}$ can be expressed in the sum of all possible TSC values τ and all the subsets $\mathcal{S}_{\vec{a},\tau}$.

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{All\ \vec{a}} \frac{1}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} \sum_{All\ \mathcal{P}_S} P(Z) \cdot \lambda(\vec{r}|\mathcal{H}, Z, \vec{1})$$

$$\sum_{All\ \mathcal{P}_S} P(Z) \cdot \lambda(\vec{r}|\mathcal{H}, Z, \vec{1})$$

$$= \frac{1}{w} \sum_{All\ \vec{a}} \frac{1}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} 2^L e^{-\beta \cdot \tau(Z)} \prod_{i=0}^{L-1} e^{-\gamma \sum_{j=0}^{U-1} |z_{i,j}|^2} \cosh\left(\sqrt{2\gamma} \cdot \text{Re}\{r_i\} \sum_{j=0}^{U-1} z_{i,j}\right)$$

$$= \frac{1}{w} \sum_{All\ \vec{a}} \frac{1}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} 2^L e^{-\beta \cdot \tau} \prod_{i=0}^{L-1} e^{-\gamma a_i^2} \cosh(\sqrt{2\gamma} \cdot \text{Re}\{r_i\} \cdot a_i)$$

$$= \sum_{All\ \vec{a}} \left(\prod_{i=0}^{L-1} e^{-\gamma \sum_{j=0}^{U-1} a_i^2} \cosh(\sqrt{2\gamma} \cdot \text{Re}\{r_i\} \cdot a_i) \right) \frac{1}{w} \left(\frac{2^L}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} e^{-\beta \cdot \tau(Z)} \right)$$

$$w = \sum_{All\ \vec{a}} \frac{2^L}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{Z \in \mathcal{S}_{\vec{a},\tau}} e^{-\beta \cdot \tau(Z)}$$

The likelihood of a CDMA signal can be expressed as a sum of a product. The first term depends only on $\vec{a}$ and can be expressed as a product of cosh functions. The second term depends on $\vec{a}$ and τ. The variable matrix Z can be replaced with the spreading matrix C. This equation is consistent with the exact likelihood calculated for simple CDMA cases as shown in Appendix F.

Theorem 1: Let $\vec{x}$ and $\vec{u}$ be vectors in $\mathbb{R}^{L \times 1}$. The average of the exponent of the dot product between $\vec{x}$ and $\vec{u}$ over the vector elements $x_i = \pm a_i$ is given by:

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} e^{\vec{u}^T \vec{x}} = \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i).$$

Proof: This property can be proven by induction. Consider vectors $\vec{x}'$ and $\vec{u}'$ be vectors in $\mathbb{R}^{L+1 \times 1}$. The dot product of vectors $\vec{x}'$ and $\vec{u}'$ can be constructed by multiplying by the exponential term $e^{u_L \cdot x_L}$ on both sides of the equation:

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} e^{\vec{u}^T \vec{x} + u_L \cdot x_L} = e^{u_L \cdot x_L} \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

Then, averaging over the variable x_L results in:

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_L = \pm a_L} e^{\vec{u}^T \vec{x} + u_L \cdot x_L} = \frac{1}{2} \sum_{x_L = \pm a_L} e^{u_L \cdot x_L} \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

$$= \cosh(e^{u_L \cdot x_L}) \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

$$= \prod_{i=0}^{L} \cosh(u_i \cdot a_i)$$

Therefore, the formula applies to vectors in $\mathbb{R}^{L+1}$.

Corollary 1-1:

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \cosh(\vec{u}^T \vec{x}) = \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

Proof: The proof is derived from Theorem 1. It requires decomposing the cosh function in a sum of exponentials.

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \cosh(\vec{u}^T \vec{x}) = \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \frac{e^{\vec{u}^T \vec{x}} + e^{-\vec{u}^T \vec{x}}}{2}$$

$$= \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \frac{e^{\vec{u}^T \vec{x}}}{2} + \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \frac{e^{-\vec{u}^T \vec{x}}}{2}$$

$$= \frac{1}{2} \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i) + \frac{1}{2} \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

$$= \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

Corollary 1-2:

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \sinh(\vec{u}^T \vec{x}) = 0$$

Proof: The proof is derived from Theorem 1. It requires decomposing the cosh function in a sum of exponentials.

$$\frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} e^{\vec{u}^T \vec{x}} = \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \cosh(\vec{u}^T \vec{x}) + \sinh(\vec{u}^T \vec{x})$$

$$= \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i) + \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \sinh(\vec{u}^T \vec{x})$$

$$= \prod_{i=0}^{L-1} \cosh(u_i \cdot a_i)$$

Therefore the average over sinh functions must equal zero.

Corollary 1-3:

$$\frac{1}{2} \sum_{y_0 \pm 1} + \cdots + \frac{1}{2} \sum_{y_{M-1} = \pm 1} \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \cosh(\vec{u}^T \vec{x}) \cosh(\vec{v}^T \vec{y})$$

$$= \frac{1}{2} \sum_{y_0 \pm 1} + \cdots + \frac{1}{2} \sum_{y_M = \pm 1} \frac{1}{2} \sum_{x_0 = \pm a_0} \cdots \frac{1}{2} \sum_{x_{L-1} = \pm a_{L-1}} \cosh(\vec{u}^T \vec{x} + \vec{v}^T \vec{y})$$

Proof: The proof is derived from Theorem 1. It requires decomposing the cosh function in a sum of exponentials.

$$\cosh(\vec{u}^T \vec{x}) \cosh(\vec{v}^T \vec{y}) = \frac{1}{4} (e^{\vec{u}^T \vec{x} + \vec{v}^T \vec{y}} + e^{\vec{u}^T \vec{x} - \vec{v}^T \vec{y}} + e^{-\vec{u}^T \vec{x} + \vec{v}^T \vec{y}} + e^{-\vec{u}^T \vec{x} - \vec{v}^T \vec{y}})$$

Averaging over $e^{\pm \vec{u}^T \vec{x} \pm \vec{v}^T \vec{y}}$ does not depend on the sign of $\vec{u}$ or $\vec{v}$. Therefore, averaging over a product of cosine functions produces four terms which averages are equal.

$$\sum_{\vec{y}} \sum_{\vec{x}} e^{\vec{u}^T \vec{x} + \vec{v}^T \vec{y}} = \sum_{\vec{y}} \sum_{\vec{x}} e^{\pm \vec{u}^T \vec{x} \pm \vec{v}^T \vec{y}}$$

$$\frac{1}{2^M}\sum_{\vec{y}}\frac{1}{2^L}\sum_{\vec{x}}\cosh(\vec{u}^T\vec{x})\cosh(\vec{v}^T\vec{y}) = \frac{1}{2^M}\sum_{\vec{y}}\frac{1}{2^L}\sum_{\vec{x}}e^{\vec{u}^T\vec{x}+\vec{v}^T\vec{y}}$$

Applying Theorem 1 and Corollary 1 to the left side of the expression proves the Corollary 3.

$$\frac{1}{2}\sum_{y_0\pm1}+\cdots+\frac{1}{2}\sum_{y_M=\pm1}\frac{1}{2}\sum_{x_0=\pm a_0}\cdots\frac{1}{2}\sum_{x_{L-1}=\pm a_{L-1}}\cosh(\vec{u}^T\vec{x})\cosh(\vec{v}^T\vec{y})$$
$$=\frac{1}{2}\sum_{y_0\pm1}+\cdots+\frac{1}{2}\sum_{y_M=\pm1}\frac{1}{2}\sum_{x_0=\pm a_0}\cdots\frac{1}{2}\sum_{x_{L-1}=\pm a_{L-1}}\cosh(\vec{u}^T\vec{x}+\vec{v}^T\vec{y})$$

APPENDIX F. Exact Likelihood
Sample Code for Simplification
of a 3x3 Hypothesis

```
(* DERIVE THE LIKELIHOOD FUNCTION *)
ClearAll[c, r, b]; Clear[CC, X, Y, α];

(* 1. DEFINE: Code Length and Number of Users *)
L = 3; U = 3;

(* 2. CONSTRUCT: CDMA Spreading Matrix *)
CC = Array[Subscript[c, #2 - 1, #1 - 1] &, {L, U}];
CC // MatrixForm
```

$$\begin{pmatrix} c_{0,0} & c_{1,0} & c_{2,0} \\ c_{0,1} & c_{1,1} & c_{2,1} \\ c_{0,2} & c_{1,2} & c_{2,2} \end{pmatrix}$$

```
(* 3. CONSTRUCT: Information Vector *)
B = Array[Subscript[b, # - 1, K] &, {U, 1}];
B // MatrixForm
```

$$\begin{pmatrix} b_{0,K} \\ b_{1,K} \\ b_{2,K} \end{pmatrix}$$

```
(* 4. CONSTRUCT: Correlator Output Formatted as a Vector *)
R = Array[Subscript[r, K, #1 - 1] &, {L, 1}];
R // MatrixForm
```

$$\begin{pmatrix} r_{K,0} \\ r_{K,1} \\ r_{K,2} \end{pmatrix}$$

```
(* 5. COMPUTE: Correlator Term of the Conditional Likelihood *)
X = R^T.CC.B ;
X = X[[1, 1]]
```

$$b_{0,K} \left(c_{0,0}\, r_{K,0} + c_{0,1}\, r_{K,1} + c_{0,2}\, r_{K,2} \right) + \\ b_{1,K} \left(c_{1,0}\, r_{K,0} + c_{1,1}\, r_{K,1} + c_{1,2}\, r_{K,2} \right) + b_{2,K} \left(c_{2,0}\, r_{K,0} + c_{2,1}\, r_{K,1} + c_{2,2}\, r_{K,2} \right)$$

```
(* 6. CONSTRUCT: Unnormalized signal model *)
S = (CC.B);
S // MatrixForm
```

$$\begin{pmatrix} b_{0,K}\, c_{0,0} + b_{1,K}\, c_{1,0} + b_{2,K}\, c_{2,0} \\ b_{0,K}\, c_{0,1} + b_{1,K}\, c_{1,1} + b_{2,K}\, c_{2,1} \\ b_{0,K}\, c_{0,2} + b_{1,K}\, c_{1,2} + b_{2,K}\, c_{2,2} \end{pmatrix}$$

```
(* 7. CONSTRUCT: Energy Term of the Likelihood Function *)
ET = (S^T.S);
ET = Expand[ET];
ET = ET /. v_^2 -> 1;
ET = ET[[1, 1]]
```

$9 + 2\, b_{0,K}\, b_{1,K}\, c_{0,0}\, c_{1,0} + 2\, b_{0,K}\, b_{1,K}\, c_{0,1}\, c_{1,1} + 2\, b_{0,K}\, b_{1,K}\, c_{0,2}\, c_{1,2} +$

$\quad 2\, b_{0,K}\, b_{2,K}\, c_{0,0}\, c_{2,0} + 2\, b_{1,K}\, b_{2,K}\, c_{1,0}\, c_{2,0} + 2\, b_{0,K}\, b_{2,K}\, c_{0,1}\, c_{2,1} +$

$\quad 2\, b_{1,K}\, b_{2,K}\, c_{1,1}\, c_{2,1} + 2\, b_{0,K}\, b_{2,K}\, c_{0,2}\, c_{2,2} + 2\, b_{1,K}\, b_{2,K}\, c_{1,2}\, c_{2,2}$

```
(* 8. CONSTRUCT: TOTAL SQUARED CORRELATION *)
C2 = CC^T.CC;
V2 = 0;
For[ i1 = 1, i1 ≤ U, i1++,
  For[ i2 = 1, i2 ≤ U, i2++,
    If[ i1 ≠ i2,
      V2 = V2 + C2[[i1, i2]]^2
    ]
  ];
];
w1 = Expand[V2] /. {_^2 → 1} /. {_^4 → 1};
FML = w1

(* 9. CONSTRUCT: Weight of the Joint Distribution of the Code
   Coefficients *)
Z = Flatten[CC];
If[ L ≠ 1,
 W1 = Exp[-β * w1];
 For[ i = 1, i ≤ Length[Z], i++,
   If[! NumberQ[Z[[i]]],
     r1 = { Z[[i]] → z };
     W1 = W1 /. r1;
     W1 = Sum[W1, {z, {-1, 1}}];
   ];
 ],
 W1 = Exp[-β];
]
```

W1

$$18 + 4\,c_{0,0}\,c_{0,1}\,c_{1,0}\,c_{1,1} + 4\,c_{0,0}\,c_{0,2}\,c_{1,0}\,c_{1,2} + 4\,c_{0,1}\,c_{0,2}\,c_{1,1}\,c_{1,2} +$$
$$4\,c_{0,0}\,c_{0,1}\,c_{2,0}\,c_{2,1} + 4\,c_{1,0}\,c_{1,1}\,c_{2,0}\,c_{2,1} + 4\,c_{0,0}\,c_{0,2}\,c_{2,0}\,c_{2,2} +$$
$$4\,c_{1,0}\,c_{1,2}\,c_{2,0}\,c_{2,2} + 4\,c_{0,1}\,c_{0,2}\,c_{2,1}\,c_{2,2} + 4\,c_{1,1}\,c_{1,2}\,c_{2,1}\,c_{2,2}$$

$$32\,e^{-54\,\beta} + 288\,e^{-22\,\beta} + 192\,e^{-6\,\beta}$$

```
(* 10.
 CONSTRUCT: Conditional likelihood given code matrix and
    information vector *)
CL = Exp[Sqrt[2 γ] * X - γ * ET - β * FML];

(* 11. COMPUTE: Average Likelihood Function *)
Clear[z];
AL = CL;
Z = Join[Flatten[CC], Flatten[B]];
For[ i = 1, i ≤ Length[Z], i++,
  If[ ! NumberQ[ Z[[i]] ],
    rl = { Z[[i]] → z };
    AL = AL /. rl;
    If[ NumberQ[Z[[i]] /. {c_,_ → 0}],
      AL = Sum[AL, { z , {-1, 1}} ],
      AL = 1 / 2 * Sum[AL, { z , {-1, 1}} ]
    ];
  ];
];

(* 11.
 SIMPLIFY: Rules to Obtain the Likelihood Function in a Simple Form *)

(* Trivial Expansion of the Exponents *)
ALS = AL /. Exp[aaa_] :→ Exp[Expand[aaa]];

(* RULE: Rename Exp[ r(i) a ] to P^a  *)
rule2 =
  { Exp[ Plus[ex1__] + ex2__ * r_{K,s_} ] :→
    Exp[Plus[ex1]] * Subscript[P, s]^Times[ex2] };
For[ i = 1, i ≤ L, i++,
 ALS = ALS /. rule2;
]
```

```
(* RULE: convert the variables P^a to Cosh[r(i)*a ] + Sinh[r(i)*a ] *)
rule3 =
  { Subscript[P, s_]^Times[ex2__] :>
    Cosh[Times[ex2] * Subscript[r, K, s]] +
      Sinh[Times[ex2] * Subscript[r, K, s] ] };
ALS = Expand[ALS /. rule3];

(* 12. FINAL LIKELIHOOD FORM  *)
ALS = Expand[ALS / W1]
```

$$\frac{24\, e^{-54\,\beta-3\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{144\, e^{-22\,\beta-3\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{48\, e^{-6\,\beta-3\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-11\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{48\, e^{-6\,\beta-11\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-11\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{48\, e^{-6\,\beta-11\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-19\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-11\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{48\, e^{-6\,\beta-11\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-19\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{24\, e^{-22\,\beta-19\,\gamma}\, \text{Cosh}\left[\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}} +$$

$$\frac{8\, e^{-54\,\beta-27\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\,\sqrt{\gamma}\, r_{K,2}\right]}{32\, e^{-54\,\beta} + 288\, e^{-22\,\beta} + 192\, e^{-6\,\beta}}$$

(* 13. CASE: When $\beta=\gamma$ *)
LF = ALS /. {$\beta \to \gamma$}

$$\frac{24\, e^{-57\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{144\, e^{-25\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{48\, e^{-9\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-33\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{48\, e^{-17\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-33\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{48\, e^{-17\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-41\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-33\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{48\, e^{-17\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-41\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{24\, e^{-41\,\gamma}\, \text{Cosh}\left[\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}} +$$

$$\frac{8\, e^{-81\,\gamma}\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,0}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,1}\right]\, \text{Cosh}\left[3\sqrt{2}\ \sqrt{\gamma}\ r_{K,2}\right]}{32\, e^{-54\,\gamma} + 288\, e^{-22\,\gamma} + 192\, e^{-6\,\gamma}}$$

```
(* 14. CONSTRUCT: EMPIRICAL LIKELIHOOD *)

(* Multicombinations *)
BZ = Floor[U / 2] + 1;
MDigitBaseBZofN = Function[{N, BZ, M}, lst = {0};
    If[ BZ > 1, lst = IntegerDigits[N, BZ]];
    If[ M > Length[lst] || Length[lst] == 0, 0, lst[[Length[lst] + 1 - M]] ] ];
```

$$\lambda = \sum_{N=0}^{BZ^L - 1} \alpha\left[L, U, \frac{1}{L} \sum_{M2=1}^{L} ((L + 1) \wedge (\text{MDigitBaseBZofN}[N, BZ, M2]) - 1)\right]$$

$$\prod_{M=1}^{L} \left(\text{Exp}[-\gamma (2*\text{MDigitBaseBZofN}[N, BZ, M] + \text{Mod}[U, 2]) \wedge 2 / 1]\right.$$

$$\left.\text{Cosh}\left[(2*\text{MDigitBaseBZofN}[N, BZ, M] + \text{Mod}[U, 2]) \sqrt{2 \gamma / 1} \; r_{K, M-1}\right]\right)$$

$e^{-3\gamma} \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 0] +$

$e^{-11\gamma} \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 1] +$

$e^{-11\gamma} \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 1] +$

$e^{-11\gamma} \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 1] +$

$e^{-19\gamma} \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 2] +$

$e^{-19\gamma} \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 2] +$

$e^{-19\gamma} \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 2] +$

$e^{-27\gamma} \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[3 \sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \alpha[3, 3, 3]$

```
(* 15. CASE: When α(β→∞) for Different Feature Vectors *)
α[3, 3, 0] =
```
$$\left(\text{ALS} //. \left\{\text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,0}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,1}\right] \text{Cosh}\left[\sqrt{2} \sqrt{\gamma} \; r_{K,2}\right] \to y, \right.\right.$$
$$\left.\left.\text{Cosh}[\_\_\_] \to 0\right\}\right) /. \{y \to 1, \gamma \to 0\}$$
```
Plot[Log[α[3, 3, 0]], {β, 0, 20}, PlotRange → Full,
 PlotLabel → "α_{3,3,0}[β]", AxesLabel → {"β", "Log[α_{3,3,0}[β]]" }]
Together[α[3, 3, 0] /. { Exp[u__ β] → z^(-u / 16) }]
```

$$\frac{24 e^{-54\beta}}{32 e^{-54\beta} + 288 e^{-22\beta} + 192 e^{-6\beta}} +$$

$$\frac{144 e^{-22\beta}}{32 e^{-54\beta} + 288 e^{-22\beta} + 192 e^{-6\beta}} + \frac{48 e^{-6\beta}}{32 e^{-54\beta} + 288 e^{-22\beta} + 192 e^{-6\beta}}$$
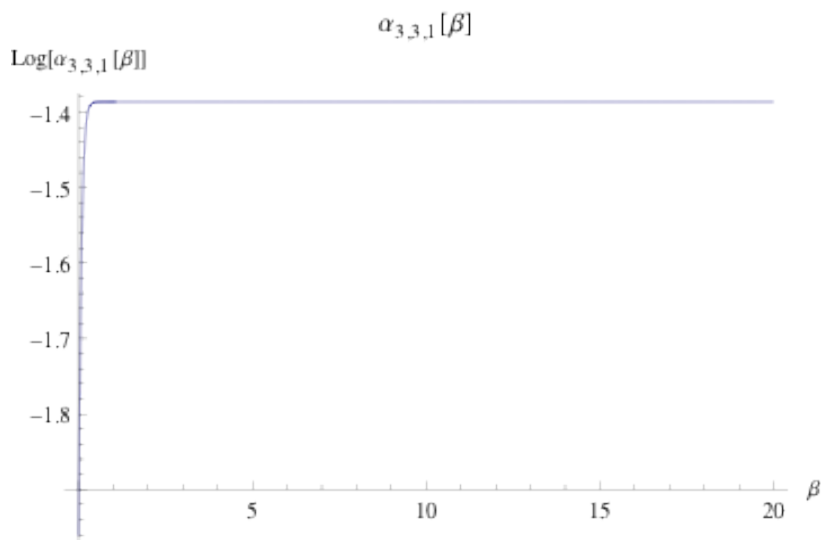
$$\alpha_{3,3,0}[\beta]$$

$$\frac{3\,(2+6\,z+z^3)}{4\,(6+9\,z+z^3)}$$

```
α[3, 3, 1] =
    (ALS //. { Cosh[3 √2 √γ r_{K,0}] Cosh[√2 √γ r_{K,1}] Cosh[√2 √γ r_{K,2}] → y,
        Cosh[___] → 0 }) /. {y → 1, γ → 0}
Plot[Log[α[3, 3, 1]], {β, 0, 20}, PlotRange → Full,
    PlotLabel → "α_{3,3,1}[β]", AxesLabel → {"β", "Log[α_{3,3,1}[β]]"}]
Together[α[3, 3, 1] /. {Exp[u__ β] → z^(-u / 16)}]
```

$$\frac{24\,e^{-22\,\beta}}{32\,e^{-54\,\beta}+288\,e^{-22\,\beta}+192\,e^{-6\,\beta}}+\frac{48\,e^{-6\,\beta}}{32\,e^{-54\,\beta}+288\,e^{-22\,\beta}+192\,e^{-6\,\beta}}$$

$$\alpha_{3,3,1}[\beta]$$

```
α[3, 3, 2] =
 Together[
   (ALS //. { Cosh[3 √2 √γ r_K,0] Cosh[3 √2 √γ r_K,1] Cosh[√2 √γ r_K,2] → y,
     Cosh[___] → 0 }) /. {y → 1, γ → 0}]
Plot[Log[α[3, 3, 2]], {β, 0, 20}, PlotRange → Full,
 PlotLabel → "α_{3,3,2}[β]", AxesLabel → {"β", "Log[α_{3,3,2}[β]]"}]
Together[α[3, 3, 2] /. { Exp[u__ β] → z^(-u / 16) }]
```

$$\frac{3\, e^{32\,\beta}}{4\,\left(1 + 9\, e^{32\,\beta} + 6\, e^{48\,\beta}\right)}$$

$\alpha_{3,3,2}[\beta]$

$$\frac{3\, z}{4\,\left(6 + 9\, z + z^3\right)}$$

```
α[3, 3, 3] =
 Together[
   (ALS //. { Cosh[3 √2 √γ r_K,0] Cosh[3 √2 √γ r_K,1] Cosh[3 √2 √γ r_K,2] → y,
     Cosh[___] → 0 }) /. {y → 1, γ → 0}]
Plot[Log[α[3, 3, 3]], {β, 0, 20}, PlotRange → Full,
 PlotLabel → "α_{3,3,3}[β]", AxesLabel → {"β", "Log[α_{3,3,3}[β]]"}]
Together[α[3, 3, 3] /. { Exp[u__ β] → z^(-u / 16) }]
```

Approved for Public Release; Distribution Unlimited

APPENDIX G. Calculation of $\alpha_{L,U,\vec{a}}(\beta)$
Using Programming Code

The implementation of the average likelihood takes the following form

$$\lambda(\vec{r}|\mathcal{H}) = \sum_{All\ \vec{a}} \alpha_{L,U,\vec{a}} \prod_{i=0}^{L-1} e^{-\gamma \sum_{j=0}^{U-1} a_i^2} \cosh(\sqrt{2\gamma} \cdot Re\{r_i\} \cdot a_i)$$

where

$$\alpha_{L,U,\vec{a}} = \frac{\frac{2^L}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{C \in S_{\vec{a},\tau}} e^{-\beta \cdot \tau(C)}}{\sum_{All\ \vec{a}} \frac{2^L}{2^{Sum(a_i=0)}} \sum_{All\ \tau} \sum_{C \in S_{\vec{a},\tau}} e^{-\beta \cdot \tau(C)}}$$

A Matlab implementation of the parameters is shown below. The code was executed using AFRL Condor HPC cluster for calculating the values of alphas when $\beta \to \infty$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculation of the numerator of the alpha parameters using formula.
% A. Vega, Last Modified: October 8, 2013
% inputs: a = vector vector, U = number of users
% output: matrix = [ TSC, occurrences ]
% formatted output = occurrences * e^(beta*TSC)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function output = alphaParam(a,U)

% Ensure positive coefficients
a = abs(a);

% Code length of the spreading matrix
L = length(s);

% Calculate possible column permutations such that sum(cols) >= 0
colPerm = [];
colPerm(1).x(1,:) = ones(1,U);
for k1 = 1:floor(U/2)
    x = ones(1,U);
    k2 = k1;
    while( k2 > 0 )
        x(1,k2) = -1;
        k2 = k2 - 1;
    end
    colPerm(k1+1).x = unique(perms(x),'rows')
end

% Display possible row vectors
if( 0 )
    for k3 = 1:size(a(:),1)
        squeeze(a(k3).x(:,:))
    end
end

% Construct matrices using row vectors
group = abs(U-a)/2+1;
elems = zeros(1,U);
for k1 = 1:L
```

```
    elems(1,k1) = size(colPerm(group(k1)).x,1);
end

% This algorithm is inefficient for L>6
tic
stack = []; mtx = zeros(L,U);
start = ones(1,L);
while( all(start<=elems) )

        % Generate Matrix
        for k1 = 1:L
            mtx(k1,:) = colPerm(group(k1)).x(start(1,k1),:);
        end

        % Calculate Total Square Correlation
        tau = 0;
        h2 = mtx'*mtx;

        for k1 = 1:L
            for k2 = 1:L
                if( k1 ~= k2 )
                    tau = tau + h2(k1,k2)^2;
                end
            end
        end

        % Store value
        stack = [stack,tau];

    %end

    % This algorithm explores all possible combinations of a row in a matrix
    % The sum of the row >= 0
    k2 = 1;
    while( k2 <= L )
        c = start(1,k2) + 1;
        if( c > elems(1,k2) )
            if( k2 < L )
                start(1,k2) = 1;
                k2 = k2 + 1;
            else
                start(1,k2) = c;
                k2 = k2 + 1;
            end
        else
            start(1,k2) = c;
            k2 = L+1;
        end
    end
end
toc

% Count all unique values
vals = unique(stack','rows')';
svals = zeros(size(vals));
for k1 = 1:length(vals)
    svals(1,k1) = sum(stack==vals(k1));
end

% Display results
[vals',svals'*2^L/2^(sum(s==0)))]
```

The coefficients α depends on the dimensions L (rows) and U (columns) of the spreading matrix C. Note that $\alpha_{L,U,\vec{a}}(\beta) = \alpha_{L,U,\vec{b}}(\beta)$ if $\vec{a}$ is a permutation of $\vec{b}$.

L = 1, U = 1

$$\alpha^{\star}_{1,1,[1]^T} = 1$$

L = 2, U = 1

$$\alpha^{\star}_{2,1,[1,1]^T} = 1$$

L = 2, U = 2

$$z = e^{-8 \cdot \beta}$$

$$\alpha_{2,2,[0,0]^T} = z/(2 * (1 + z))$$

$$\alpha^{\star}_{2,2,[2,0]^T} = 1/(2 * (1 + z))$$

$$\alpha_{2,2,[2,2]^T} = z/(2 * (1 + z))$$

L = 3, U = 1

$$\alpha^{\star}_{3,1,[1,1,1]^T} = 1$$

L = 3, U = 2

$$z = e^{-16 \cdot \beta}$$

$$\alpha_{3,2,[0,0,0]^T} = z/(2 * (3 + z))$$

$$\alpha^{\star}_{3,2,[2,0,0]^T} = 1/(2 * (3 + z))$$

$$\alpha^{\star}_{3,2,[2,2,0]^T} = 1/(2 * (3 + z))$$

$$\alpha_{3,3,[2,2,2]^T} = z/(2 * (3 + z))$$

L = 3, U = 3

$$z = e^{-16 \cdot \beta}$$

$$\alpha^{\star}_{3,2,[1,1,1]^T} = 3 * (2 + 6z + z^3)/(4 * (6 + 9z + z^3))$$

$$\alpha^{\star}_{3,2,[3,1,1]^T} = 3 * (2 + z)/(4 * (6 + 9z + z^3))$$

$$\alpha_{3,3,[3,3,1]^T} = 3z/(4 * (6 + 9z + z^3))$$

$$\alpha_{3,3,[3,3,3]^T} = z^3/(4 * (6 + 9z + z^3))$$

L = 4, U = 1

$$\alpha^\star_{4,1,[1,1,1,1]^T} = 1$$

L = 4, U = 2

$$z = e^{-8\cdot\beta}$$

$$\alpha_{4,2,[0,0,0,0]^T} = z^4/(2 * (1 + z)^2 * (3 - 2z + z^2))$$

$$\alpha_{4,2,[2,0,0,0]^T} = z/(2 * (1 + z)^2 * (3 - 2z + z^2))$$

$$\alpha^\star_{4,2,[2,2,0,0]^T} = 1/(2 * (1 + z)^2 * (3 - 2z + z^2))$$

$$\alpha_{4,2,[2,2,2,0]^T} = z/(2 * (1 + z)^2 * (3 - 2z + z^2))$$

$$\alpha_{4,2,[2,2,2,2]^T4} = z^4/(2 * (1 + z)^2 * (3 - 2z + z^2))$$

L = 4, U = 3

$$z = e^{-8\cdot\beta}$$

$$\alpha_{4,3,[1,1,1,1]^T} = 3z * (12 + 6z + 8z^2 + z^5)/(4 * (6 + 36z + 9z^2 + 12z^3 + z^6))$$

$$\alpha^\star_{4,3,[3,1,1,1]^T} = 3 * (2 + 6z + z^3)/(4 * (6 + 36z + 9z^2 + 12z^3 + z^6))$$

$$\alpha_{4,3,[3,3,1,1]^T} = 3z * (2 + z)/(4 * (6 + 36z + 9z^2 + 12z^3 + z^6))$$

$$\alpha_{4,3,[3,3,3,1]^T} = 3z^3/(4 * (6 + 36z + 9z^2 + 12z^3 + z^6))$$

$$\alpha_{4,3,[3,3,3,3]^T} = z^6/(4 * (6 + 36z + 9z^2 + 12z^3 + z^6))$$

L = 4, U = 4

$$z = e^{-8\cdot\beta}$$

$$q(z) = (8 * (6 + 96z^3 + 144z^4 + 144z^7 + 81z^8 + 24z^{12} + 16z^{15} + z^{24}))$$

$$\alpha_{4,4,[0,0,0,0]^T} = 3z^4 * (12 + 6z^4 + 8z^8 + z^{20})/q(z)$$

$$\alpha_{4,4,[2,0,0,0]^T} = 12z^3 * (2 + 6z^4 + z^{12})/q(z)$$

$$\alpha_{4,4,[2,2,0,0]^T} = 12z^4 * (6 + 5z^4 + z^8)/q(z)$$

$$\alpha_{4,4,[2,2,2,0]^T} = 12z^3 * (6 + 9z^4 + z^{12})/q(z)$$

$$\alpha^{\star}_{4,4,[2,2,2,2]^T} = 4 * (6 + 36z^4 + 9z^8 + 12z^{12} + z^{24})/q(z)$$

$$\alpha^{\star}_{4,4,[4,0,0,0]^T} = 3 * (2 + 6z^4 + z^{12})/q(z)$$

$$\alpha_{4,4,[4,2,0,0]^T} = 12z^3 * (2 + z^4)/q(z)$$

$$\alpha_{4,4,[4,2,2,0]^T} = 12z^4 * (3 + z^4)/q(z)$$

$$\alpha_{4,4,[4,2,2,2]^T} = 4z^3 * (6 + 9z^4 + z^{12})/q(z)$$

$$\alpha_{4,4,[4,4,0,0]^T} = 3z^4 * (2 + z^4)/q(z)$$

$$\alpha_{4,4,[4,4,2,0]^T} = 12z^7/q(z)$$

$$\alpha_{4,4,[4,4,2,2]^T} = 4z^8 * (3 + z^4)/q(z)$$

$$\alpha_{4,4,[4,4,4,0]^T} = 3z^{12}/q(z)$$

$$\alpha_{4,4,[4,4,4,2]^T} = 4z^{15}/q(z)$$

$$\alpha_{4,4,[4,4,4,4]^T} = z^{24}/q(z)$$

We define the parameter $\alpha_{L,U,\vec{a}}(\beta)$ using:

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}e^{-\beta\cdot\tau}}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}e^{-\beta\cdot\tau}}$$

Proposition 5.   The limit of $\alpha_{L,U,\vec{a}}(\beta)$ as β approaches infinity depends on the feature vectors as follows:

$$\lim_{\beta\to\infty}\alpha_{L,U,\vec{a}}(\beta) = \begin{cases} 0 \ for\ \vec{a}\neq\vec{a}^{\star} \\ \dfrac{\frac{2^L}{2^{Sum(a_i=0)}}\left|S_{\vec{a},\tau_{min}}\right|}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Sum(a_i=0)}}\left|S_{\vec{a},\tau_{min}}\right|} > 0 \ for\ \vec{a}=\vec{a}^{\star}\end{cases}$$

Proof: The expression can be converted in a polynomial rational function by substituting $e^{-\beta}\to y$ and divide by the exponent with the minimum Total Square Correlation $y^{\tau_{min}}$ in both side of the equation.

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}y^{\tau}}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}y^{\tau}}$$

$$= \frac{\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}y^{\tau-\tau_{min}}}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Sum(a_i=0)}}\sum_{All\ \tau}\sum_{Z\in S_{\vec{a},\tau}}y^{\tau-\tau_{min}}}$$

$$\tau = \tau(Z) \geq \tau_{min} \geq 0$$

$$\vec{a} = abs(Z\cdot\vec{1})$$

Since $\tau - \tau_{min} \geq 0$, taking the limit:

$$\lim_{y\to 0}y^{\tau-\tau_{min}} = \begin{cases} 0\ for\ \tau\neq\tau_{min} \\ 1\ for\ \tau=\tau_{min}\end{cases}$$

eliminates all terms with Total Square Correlation terms greater than τ_{min}.   The result is:

$$\alpha_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Sum(a_i=0)}}\sum_{Z\in S_{\vec{a},\tau_{min}}}1}{\sum_{All\ \vec{a}}\frac{2^L}{2^{Sum(a_i=0)}}\sum_{Z\in S_{\vec{a},\tau_{min}}}1} \geq 0$$

The summation over all the terms in the set is just the cardinality of the set, therefore:

$$\alpha^{\star}_{L,U,\vec{a}}(\beta) = \frac{\frac{2^L}{2^{Sum(a_i=0)}}\left| \mathcal{S}_{\vec{a},\tau_{min}} \right|}{\sum_{All\ \vec{a}} \frac{2^L}{2^{Sum(a_i=0)}}\left| \mathcal{S}_{\vec{a},\tau_{min}} \right|}.$$

Table J-1. Several Feature Vectors

	L=2	L=3	L=4	L=5	L=6
U=1	$[1,1]^T$	$[1,1,1]^T$	$[1,1,1,1]^T$	$[1,1,1,1,1]^T$	$[1,1,1,1,1,1]^T$
U=2	$[2,0]^T$	$[2,0,0]^T$ $[2,2,0]^T$	$[2,2,0,0]^T$	$[2,2,0,0,0]^T$ $[2,2,2,0,0]^T$	$[2,2,2,0,0,0]^T$
U=3		$[3,1,1]^T$ $[1,1,1]^T$	$[3,3,1,1]^T$	$[3,3,1,1,1]^T$ $[3,3,3,1,1]^T$	$[3,1,1,1,1,1]^T$ $[3,3,1,1,1,1]^T$
U=4			$[2,2,2,2]^T$ $[4,0,0,0]^T$	$[2,2,2,2,2]^T$ $[4,0,0,0,0]^T$ $[2,2,2,2,0]^T$ $[4,2,0,0,0]^T$ $[4,2,2,2,2]^T$ $[4,4,0,0,0]^T$	not calculated
U=5				$[3,3,3,3,3]^T$ $[5,1,1,1,1]^T$ $[3,3,1,1,1]^T$	not calculated
U=6					$[2,2,2,0,0,0]^T$ $[4,2,2,2,0,0]^T$

The Corresponding Lowest TSC Spreading Matrices
Case: $L = 2, U = 2, \tau_{min} = 0$

$$C^\star = \begin{bmatrix} + & + \\ + & - \end{bmatrix}$$

Case: $L = 3, U = 2, \tau_{min} = 2$

$$C^\star = \begin{bmatrix} + & + \\ + & - \\ + & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & - \\ - & + \end{bmatrix}$$

Case: $L = 3, U = 3, \tau_{min} = 6$

$$C^\star = \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & - \end{bmatrix}, \begin{bmatrix} + & + & - \\ + & - & + \\ - & + & + \end{bmatrix}$$

Case: $L = 4, U = 2, \tau_{min} = 0$

$$C^\star = \begin{bmatrix} + & + \\ + & - \\ + & + \\ + & - \end{bmatrix}$$

Case: $L = 4, U = 3, \tau_{min} = 0$

$$C^\star = \begin{bmatrix} + & + & + \\ + & - & + \\ - & + & + \\ + & + & - \end{bmatrix}$$

Case: $L = 4, U = 4, \tau_{min} = 0$

$$C^\star = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & + & - & + \end{bmatrix}, \begin{bmatrix} + & + & - & + \\ + & - & + & + \\ - & + & + & + \\ + & + & + & - \end{bmatrix}$$

Case: $L = 5, U = 2, \tau_{min} = 2$

$$C^\star = \begin{bmatrix} + & + \\ + & - \\ + & + \\ + & - \\ + & + \end{bmatrix}$$

Case: $L = 5, U = 3, \tau_{min} = 6$

$$C^\star = \begin{bmatrix} - & + & + \\ + & + & + \\ + & - & + \\ + & + & - \\ + & - & + \end{bmatrix} \begin{bmatrix} + & + & + \\ + & + & - \\ - & - & + \\ + & + & + \\ + & + & + \end{bmatrix}$$

Case: $L = 5, U = 4, \tau_{min} = 12$

$$C^\star = \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & - & + \\ + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & - & + \\ + & - & + & + \\ - & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}, \begin{bmatrix} + & + & - & + \\ + & - & + & + \\ - & + & + & + \\ + & + & + & - \\ + & + & + & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{bmatrix}$$

Case: $L = 5, U = 5, \tau_{min} = 20$

$$C^\star = \begin{bmatrix} + & - & + & + & + \\ - & + & + & + & + \\ + & + & - & + & - \\ + & + & + & - & - \\ + & + & - & - & + \end{bmatrix}, \begin{bmatrix} + & + & - & + & + \\ + & - & + & + & + \\ - & + & + & + & + \\ + & + & + & + & + \\ + & + & + & + & - \end{bmatrix}, \begin{bmatrix} + & + & + & + & + \\ - & + & + & - & + \\ - & + & - & + & + \\ - & - & + & + & + \\ - & + & + & + & - \end{bmatrix},$$

APPENDIX K. Simplification Using Sylvester's Construction

The average likelihood function under the assumption of a Sylvester's construction is:

$$\lambda(\vec{r}|\mathcal{H} = H_L) = \left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \cdot \sum_{\substack{\text{All permutations} \\ \text{of } \vec{\xi}^\star \text{ elements}}} \prod_{i=0}^{L-1} e^{-\gamma|\xi_i|^2} \cosh\left(\sqrt{2\gamma} \cdot \xi_i \cdot r_i\right).$$

The likelihood ratio between hypotheses $\mathcal{H}_{2L} = H_{2L}$ and $\mathcal{H}_L = H_L$ will be considered:

$$\lambda(\vec{r}|\mathcal{H} = H_{2L}) = \left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^{2L}}\right)^{\frac{1}{2}} \cdot \sum_{k=0}^{2L-1} e^{-\gamma|2L|^2} \cosh\left(\sqrt{2\gamma} \cdot 2L \cdot r_i\right)$$

and

$$\lambda(\vec{r}|\mathcal{H} = H_L) = \left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \cdot \sum_{k=0}^{L-1} e^{-\gamma|L|^2} \cosh\left(\sqrt{2\gamma} \cdot L \cdot r_i\right)$$

For each sample vector $\vec{r}_{2L\times 1}$ in the $\mathcal{H}_{2L}$ there are two sample vectors $\vec{r}_{L\times 1}$ in $\mathcal{H}_L$. These two vectors are statistically independent, so the likelihood in the denominator is multiplied.

$$\Lambda(r_0, r_1, \ldots, r_{2L}) = \frac{\lambda(r_0, r_1, \ldots, r_{2L}|\mathcal{H} = H_{2L})}{\lambda(r_0, r_1, \ldots, r_{L-1}|\mathcal{H} = H_L) \cdot \lambda(r_L, r_{L+1}, \ldots, r_{2L}|\mathcal{H} = H_L)}$$

Substituding in the previous equation gives:

$$\Lambda(r_0, r_1, \ldots, r_{2L})$$

$$= \frac{\left(\frac{e^{-\langle\vec{r},\vec{r}\rangle}}{(\pi N_0)^{2L}}\right)^{\frac{1}{2}} \cdot \sum_{k=0}^{L-1} e^{-\gamma|2L|^2} \cosh\left(\sqrt{2\gamma} \cdot 2L \cdot r_i\right)}{\left(\frac{e^{-\langle\vec{r}_1,\vec{r}_1\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \cdot \sum_{k=0}^{L-1} e^{-\gamma|L|^2} \cosh\left(\sqrt{2\gamma} \cdot L \cdot r_i\right) \cdot \left(\frac{e^{-\langle\vec{r}_2,\vec{r}_2\rangle}}{(\pi N_0)^L}\right)^{\frac{1}{2}} \cdot \sum_{k=0}^{L-1} e^{-\gamma|L|^2} \cosh\left(\sqrt{2\gamma} \cdot L \cdot r_{i+L}\right)}$$

Note that $\langle \vec{r}_1, \vec{r}_1 \rangle$ and $\langle \vec{r}_2, \vec{r}_2 \rangle$ is the dot product between the first and second sample of $\mathcal{H}_L$. The sum of both dot products equals the dot product in the numerator.

$$\langle \vec{r}_1, \vec{r}_1 \rangle + \langle \vec{r}_2, \vec{r}_2 \rangle = \langle \vec{r}, \vec{r} \rangle$$

So, the product between

$$\left(\frac{e^{-\langle \vec{r}_1, \vec{r}_1 \rangle}}{(\pi N_0)^L} \right)^{\frac{1}{2}} \left(\frac{e^{-\langle \vec{r}_2, \vec{r}_2 \rangle}}{(\pi N_0)^L} \right)^{\frac{1}{2}} = \left(\frac{e^{-\langle \vec{r}, \vec{r} \rangle}}{(\pi N_0)^{2L}} \right)^{\frac{1}{2}}$$

cancels with the same term in the numerator.

The final form is given by:

$$\frac{\lambda(\vec{r}|\mathcal{H}_{2L})}{\lambda(\vec{r}|\mathcal{H}_L)} = \frac{\sum_{k=0}^{2L-1} \cosh(\sqrt{2\gamma} \cdot 2L \cdot Re\{r_k\})}{\sum_{k=0}^{L-1} \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\}) \cdot \sum_{k=L+1}^{2L-1} \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\})}$$

Using the double angle formulas for cosh functions helps reducing the amount of calculations by reusing the terms in the denominator.

$$\frac{\lambda(\vec{r}|\mathcal{H}_{2L})}{\lambda(\vec{r}|\mathcal{H}_L)} = \frac{\sum_{k=0}^{2L}(2d_k^2 - 1)}{\sum_{k=0}^{L} d_k \cdot \sum_{k=L+1}^{2L} d_k}$$

$$d_k = \cosh(\sqrt{2\gamma} \cdot L \cdot Re\{r_k\})$$

Note: It was discovered that the likelihood ratio is only valid when the energy of the signal is assumed to be known. A more correct expression will consider the effect of the dependency of the hypothesis on the energy of the CDMA signal. In other words, d_k is a function of $\mathcal{H}$.

List of Acronyms

ALRT	Average Likelihood Ratio Test
CDMA	Code Division Multiple Access
LRT	Likelihood Ratio Test
MPI	Multiple Programming Interface
$Re\{\}$	Real part
TSC, τ	Total Square Correlation
$\otimes$	Kronecker product
$\vec{c}_{*,i}$	Column vector
γ	Chip Signal to Noise Ratio
γ_s	Symbol Signal to Noise Ratio
$\Lambda_{2L}(\vec{r})$	Likelihood ratio (Hypothesis 2L versus L)
E	Average energy per symbol
$\lambda(\vec{r}|\mathcal{H})$	Likelihood function
$\lambda(\vec{r}|\mathcal{H}, C, \vec{b})$	Conditional likelihood function
C	Spreading matrix
$C^\star$	TSC optimal spreading matrix
$\vec{b}$	Information vector
$\alpha_{L,U,\vec{a}}$	Coefficients of the CDMA likelihood function
$\alpha^\star$	Non-vanishing alpha coefficients
$\vec{a}^\star$	Feature vector for CDMA detection